WAKE FIELDS AND IMPEDANCES

- **Wake fields (15 slides)**
- **Impedances (8)**
- **Generalized notion of impedance for asymmetric structures (24)**
  - **Dipolar and quadrupolar transverse impedances (and more)**
  - **1-wire and 2-wire bench measurements**
  - **Yokoya factors for dipolar and quadrupolar impedances**
- **Impedance of an infinitely long smooth beam pipe (31)**
- **Impedance and wake potential of a resonator (25)**
- **Cut-off frequencies in a circular waveguide (7)**
- **Examples of ElectroMagnetic simulations (17)**
  - **Example from CST => Wake field simulation of a collimator**
  - **A tertiary LHC collimator chamber with the HFSS code**
  - **A LHC graphite collimator with the HFSS code**
  - **The CMS vacuum chamber (in the LHC) with ABCI code**

WAKE FIELDS (1/15)

- A beam of charged particles move around an accelerator under the Lorentz force produced by the "external" electromagnetic fields (from the guiding and focusing magnets, RF cavities etc.)

$$\vec{F}_{ext} = e\left(\vec{E}_{ext} + \vec{\upsilon} \times \vec{B}_{ext} \right)$$

- However, the charged particles also interact with their environment, inducing image charges and currents which create electromagnetic fields called "WAKE FIELDS"

Perturbation proportional to the beam intensity

$$\vec{F}_{wake} = e\left(\vec{E}_{wake} + \vec{\upsilon} \times \vec{B}_{wake} \right)$$

- Therefore, the motion of the charged particles should be computed considering these "perturbations"

WAKE FIELDS (2/15)

- The 2 fundamental approximations

1) The rigid-beam approximation

⇒ The beam traverses a piece of equipment rigidly, i.e. the wake-field perturbation does not affect the motion of the beam during the traversal of the impedance

⇒ The distance z of the test particle behind some source particle does not change

2) The impulse approximation

⇒ As the test particle moves at the fixed velocity v = β c through a piece of equipment, what is important is the impulse (and not the force)

$$\Delta \vec{p}(x,y,z) = \int_{-\infty}^{+\infty} dt\ \vec{F}(x,y,s=z+\beta c\,t,t) = \int_{-\infty}^{+\infty} dt\ e\left(\vec{E}+\vec{\upsilon}\times\vec{B}\right)$$

WAKE FIELDS (3/15)

- **Position of the source particle** $s_{source} = \upsilon\, t$

- **Position of the test particle** $\boxed{s_{test} = s_{source} + z = \upsilon\, t + z}$

z < 0 and time-independent

- **Maxwell equations for a particle in the beam**

$$\vec{\nabla} \,.\, \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla} \,.\, \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0\, \rho\, \upsilon\, \vec{s} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

WAKE FIELDS (4/15)

- **Lorentz force** $\vec{F} = e\left(\vec{E} + \upsilon\,\vec{s} \times \vec{B}\right)$

- **Using** $\vec{\nabla} .\left(\vec{A} \times \vec{B}\right) = \vec{B} .\left(\vec{\nabla} \times \vec{A}\right) - \vec{A} .\left(\vec{\nabla} \times \vec{B}\right)$ **, it yields**

=> $$\vec{\nabla} .\vec{F} = e\left[\frac{\rho}{\varepsilon_0} - \upsilon\,\vec{s} .\left(\mu_0\,\rho\,\upsilon\,\vec{s} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}\right)\right]$$

=> $$\boxed{\vec{\nabla} .\vec{F} = \frac{e\,\rho}{\varepsilon_0\,\gamma^2} - \frac{e\,\beta}{c}\frac{\partial E_s}{\partial t}}$$

- **Using** $\vec{\nabla} \times \left(\vec{A} \times \vec{B}\right) = \vec{A}\left(\vec{\nabla} .\vec{B}\right) - \vec{B} .\left(\vec{\nabla} .\vec{A}\right) + \left(\vec{B} .\vec{\nabla}\right)\vec{A} - \left(\vec{A} .\vec{\nabla}\right)\vec{B}$ **, it yields**

WAKE FIELDS (5/15)

$$\boxed{\vec{\nabla} \times \vec{F} = - e \left(\frac{\partial}{\partial t} + \beta c \frac{\partial}{\partial s} \right) \vec{B}}$$

- **Applying now the curl to the impulse, gives**

$$\vec{\nabla} \times \Delta \vec{p}(x, y, z) = \int_{-\infty}^{+\infty} dt \left[\vec{\nabla} \times \vec{F}(x, y, s = z + \beta c t, t) \right]$$

=>

$$\vec{\nabla} \times \Delta \vec{p}(x, y, z) = - e \int_{-\infty}^{+\infty} dt \left(\frac{\partial}{\partial t} + \beta c \frac{\partial}{\partial s} \right) \vec{B}(x, y, s = z + \beta c t, t)$$

=>

$$\vec{\nabla} \times \Delta \vec{p}(x, y, z) = - e \int_{-\infty}^{+\infty} dt \frac{d \vec{B}}{d t} = - e \left[\vec{B}(x, y, s = z + \beta c t, t) \right]_{t = -\infty}^{t = +\infty} = 0$$

WAKE FIELDS (6/15)

- This relation is known as the Panofsky-Wenzel theorem

$$\boxed{\vec{\nabla} \times \Delta \vec{p}(x, y, z) = 0}$$

For β = constant

- It is very general as:
  - No boundary conditions have been imposed so far
  - Only the 2 fundamental approximations have been made
    - Rigid bunch
    - Impulse
  - β should be constant and does not need to be 1

- Another important relation can be obtained when $\boxed{\beta = 1}$, taking the divergence of the impulse

WAKE FIELDS (7/15)

$$\vec{\nabla}\,.\Delta\vec{p}(x,y,z) = \int_{-\infty}^{+\infty} dt \left[\frac{e\,\rho}{\varepsilon_0\,\gamma^2} - \frac{e\,\beta}{c}\,\frac{\partial E_s}{\partial t}\right] = -\frac{e}{c}\int_{-\infty}^{+\infty} dt\,\frac{\partial E_s(x,y,s=z+\beta c\,t,t)}{\partial t}$$

Furthermore,

$$\frac{d\,E_s}{d\,t} = \frac{\partial E_s}{\partial t} + \frac{\partial E_s}{\partial s}\,\frac{d\,s}{d\,t} = \frac{\partial E_s}{\partial t} + \frac{\partial E_s}{\partial s}\,c$$

=>

$$\vec{\nabla}\,.\Delta\vec{p}(x,y,z) = -\frac{e}{c}\int_{-\infty}^{+\infty} dt\left[\frac{d\,E_s}{d\,t} - \frac{\partial E_s}{\partial s}\,c\right]$$

$$\left[E_s\right]_{t=-\infty}^{t=+\infty} = 0$$

=>

$$\vec{\nabla}\,.\Delta\vec{p}(x,y,z) = e\int_{-\infty}^{+\infty} dt\,\frac{\partial E_s(x,y,s=z+\beta c\,t,t)}{\partial s} = \frac{\partial}{\partial s}\left[e\int_{-\infty}^{+\infty} dt\;E_s(x,y,s=z+\beta c\,t,t)\right]$$

WAKE FIELDS (8/15)

=> $\vec{\nabla}\,.\Delta\,\vec{p}\left(x,y,z\right)=\dfrac{\partial\,\Delta p_s}{\partial\,s}$ => $\vec{\nabla}_\perp\,.\Delta\,\vec{p}_\perp=0$ For β = 1

- Considering the case of a cylindrically symmetric chamber (using cylindrical coordinates $r\,,\vartheta\,,z$), yields the following 3 equations from Panofsky-Wenzel theorem

$$\frac{1}{r}\left(\frac{\partial\,\Delta p_z}{\partial\,\vartheta}\right)=\frac{\partial\,\Delta p_\vartheta}{\partial\,z}$$

$$\frac{\partial\,\Delta p_r}{\partial\,z}=\frac{\partial\,\Delta p_z}{\partial\,r}$$

$$\frac{\partial\left(r\,\Delta p_\vartheta\right)}{\partial\,r}=\frac{\partial\,\Delta p_r}{\partial\theta}$$

+ a 4th relation, when $\beta=1$, $\dfrac{\partial\left(r\,\Delta p_r\right)}{\partial\,r}=-\dfrac{\partial\,\Delta p_\theta}{\partial\theta}$

WAKE FIELDS (9/15)

- We will consider the following source charge density. A macro-particle of charge $Q = N_b \, e$ is assumed to move along the pipe (in the s-direction) with an offset $r = a$ in the $\vartheta = 0$ direction and with velocity $\upsilon = \beta \, c$ (equal to the bunch velocity)

=>

$$\boxed{\begin{aligned} \rho(r, \vartheta, s\,; t) &= \frac{Q}{a} \, \delta(r - a) \, \delta_p(\vartheta) \, \delta(s - \upsilon \, t) \\ &= \sum_{m=0}^{\infty} \frac{Q_m \cos(m \vartheta)}{\pi \, a^{m+1} \left(1 + \delta_{m0}\right)} \, \delta(r - a) \, \delta(s - \upsilon \, t) = \sum_{m=0}^{\infty} \rho_m \end{aligned}}$$

$$\boxed{Q = N_b \, e} \qquad \boxed{Q_m = Q \, a^m}$$

using the relation

$$T \, \delta_p(\vartheta) = T \sum_{k=-\infty}^{k=+\infty} \delta(\vartheta - k \, T) = \sum_{m=-\infty}^{m=+\infty} e^{\, j \, m \, 2 \pi \frac{\vartheta}{T}}$$

WAKE FIELDS (10/15)

In frequency domain it gives

$$\rho(r,\vartheta,s;\omega)=\sum_{m=0}^{\infty}\frac{Q_m\cos(m\vartheta)}{\upsilon\,\pi\,a^{m+1}\left(1+\delta_{m0}\right)}\,\delta(r-a)\,e^{-jks}$$

$$k=\frac{\omega}{\upsilon}$$

$$\vec{J}(r,\vartheta,s;\omega)=\rho(r,\vartheta,s;\omega)\,\vec{\upsilon}=\sum_{m=0}^{\infty}\vec{J}_m=\sum_{m=0}^{\infty}\rho_m\,\vec{\upsilon}$$

using the relation

$$\delta(s-\upsilon\,t)=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,e^{j\omega t}\left[\frac{e^{-jks}}{\upsilon}\right]$$

WAKE FIELDS (11/15)

- Looking at the longitudinal electric field (Maxwell equation) yields

$$\Delta p_z = \Delta \hat{p}_z \cos m\theta$$

=> (from the previous equations) $\Delta p_r = \Delta \hat{p}_r \cos m\theta$ $\Delta p_\theta = \Delta \hat{p}_\theta \sin m\theta$

and the 4 equations become

$$-\frac{m}{r} \Delta \hat{p}_z = \frac{\partial \Delta \hat{p}_\vartheta}{\partial z}$$

$$\frac{\partial \Delta \hat{p}_r}{\partial z} = \frac{\partial \Delta \hat{p}_z}{\partial r}$$

$$\frac{\partial \left(r \Delta \hat{p}_\vartheta \right)}{\partial r} = -m \Delta \hat{p}_r$$

$$\frac{\partial \left(r \Delta \hat{p}_r \right)}{\partial r} = -m \Delta \hat{p}_\theta$$

WAKE FIELDS (12/15)

- For m = 0, $\Delta\hat{p}_r = \Delta\hat{p}_\theta = 0$, otherwise the 3rd and 4th equations would give a term inversely proportional to *r*, which is singular at 0

- For m ≠ 0, the 3rd and 4th equations give $\frac{\partial}{\partial r}\left[r \frac{\partial \left(r \Delta\hat{p}_r \right)}{\partial r} \right] = m^2 \, \Delta\hat{p}_r$

=> $\Delta p_r \left(r,\theta,z \right) \propto r^{m-1} \cos m\theta$

q (*Q*) is the charge of the test (source) particle

- The whole solution can be written as, for m ≥ 0,

$$\upsilon \, \Delta p_s \left(r,\theta,z \right) = \int_0^L F_s \, ds = - q \, Q \, a^m \; r^m \cos m\theta \;\; W'_m \left(z \right)$$

$$\upsilon \, \Delta p_r \left(r,\theta,z \right) = \int_0^L F_r \, ds = - q \, Q \, a^m \;\; m \, r^{m-1} \cos m\theta \;\; W_m \left(z \right)$$

$$\upsilon \, \Delta p_\theta \left(r,\theta,z \right) = \int_0^L F_\theta \, ds = q \, Q \, a^m \;\; m \, r^{m-1} \sin m\theta \;\; W_m \left(z \right)$$

WAKE FIELDS (13/15)

- $W_m(z)$ is called the transverse wake function of azimuthal mode *m* and $W'_m(z)$ is called the longitudinal wake function of azimuthal mode *m*

- They describe the shock response of the vacuum chamber environment to a δ–function beam which carries an mth moment

- Mathematically, $W_m(z)$ resembles a Green's function

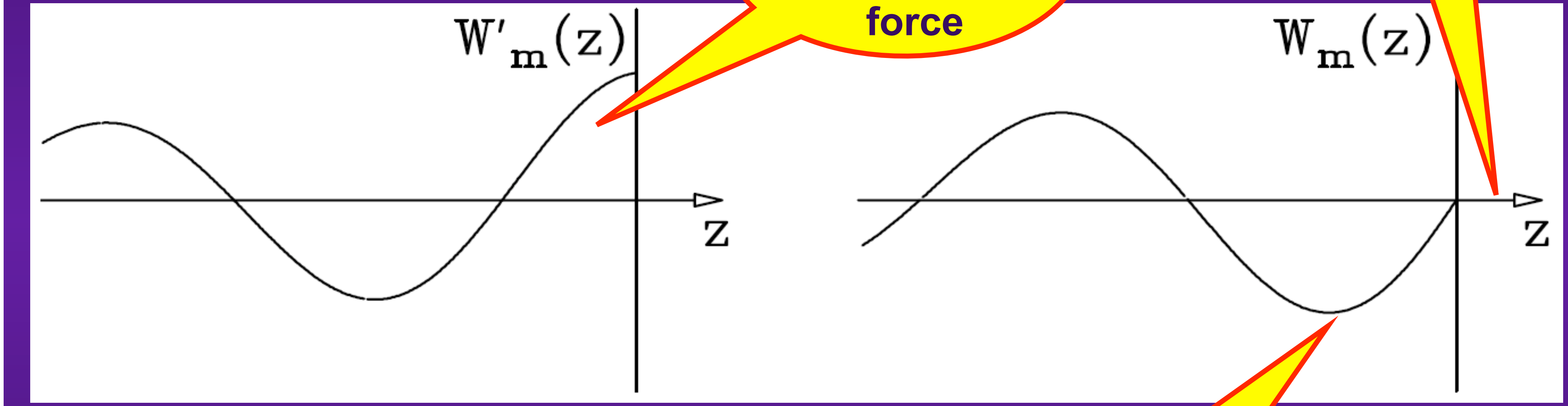
- The integrals (on the left) are called wake potentials

- Longitudinal wake function for *m* = 0 and transverse wake function for *m* = 1

$$W'_0(z) = -\frac{1}{q\,Q}\int_0^L F_s\,ds = -\frac{1}{Q}\int_0^L E_s\,ds$$

$$W_1(z) = -\frac{1}{q\,Q\,a}\int_0^L F_x\,ds = -\frac{1}{Q\,a}\int_0^L \left(E_x - \upsilon\,B_y\right) ds$$

WAKE FIELDS (14/15)



From causality (for v = c)

> 0 just after the source => Decelerating force

< 0 just after the source => Same direction of the deflection of the source

WAKE FIELDS (15/15)

$\mathrm{N = m\,kg\,s^{-2}}$

$\mathrm{V = m^2\,kg\,s^{-3}\,A^{-1}}$

$\mathrm{C = A\,s}$

- **Units of the wake fields**

$$W_0'(z) = -\frac{\upsilon\,\Delta p_s}{q\,Q} \rightarrow \frac{\mathrm{N\,m}}{C^2} = \frac{\mathrm{m\,kg\,s^{-2}\,m}}{C^2} = \frac{\mathrm{V}}{\mathrm{C}}$$

$$W_m'(z) \rightarrow \frac{\mathrm{V}}{\mathrm{C\,m}^{2m}}$$

$$W_1(z) = -\frac{\upsilon\,\Delta p_r}{q\,Q\,a} \rightarrow \frac{\mathrm{V}}{\mathrm{C\,m}}$$

$$W_m(z) \rightarrow \frac{\mathrm{V}}{\mathrm{C\,m}^{2m-1}}$$

- **Some comments on the wake fields**
  - It is here for cylindrically symmetric structures => More involved for asymmetric structures (e.g. quadrupolar wake field)
  - More involved when $\beta \neq 1$, as in this case there are also some fields in front of the source particle

IMPEDANCES (1/8)

- **The impedances are related to the wake functions by Fourier transforms**

$$Z_m^{//}(\omega) \rightarrow \frac{\mathrm{V}}{\mathrm{C}\,\mathrm{m}^{2m}} \times \mathrm{s} = \frac{\Omega}{\mathrm{m}^{2m}}$$

$$Z_m^{\perp}(\omega) \rightarrow \frac{\mathrm{V}}{\mathrm{C}\,\mathrm{m}^{2m-1}} \times \mathrm{s} = \frac{\Omega}{\mathrm{m}^{2m-1}}$$

$$\boxed{Z_m^{//}(\omega) = -\int_{-\infty}^{+\infty} W_m'(z)\, e^{jkz}\, \frac{dz}{\upsilon} = \int_{-\infty}^{+\infty} W_m'(t)\, e^{jks}\, e^{-j\omega t}\, dt}$$

$$\boxed{Z_m^{\perp}(\omega) = j \int_{-\infty}^{+\infty} W_m(z)\, e^{jkz}\, \frac{dz}{\upsilon} = -\, j \int_{-\infty}^{+\infty} W_m(t)\, e^{jks}\, e^{-j\omega t}\, dt}$$

$$\boxed{W_m'(z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Z_m^{//}(\omega)\, e^{-jkz}\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Z_m^{//}(\omega)\, e^{-jks}\, e^{j\omega t}\, d\omega}$$

$$\boxed{W_m(z) = \frac{j}{2\pi} \int_{-\infty}^{+\infty} Z_m^{\perp}(\omega)\, e^{-jkz}\, d\omega = \frac{j}{2\pi} \int_{-\infty}^{+\infty} Z_m^{\perp}(\omega)\, e^{-jks}\, e^{j\omega t}\, d\omega}$$

IMPEDANCES (2/8)

- 2 important properties of the impedances
  - As the wake functions are real, it can be shown that

$$\boxed{\left[Z_m^{//}(\omega) \right]^* = Z_m^{//}(-\omega)}$$

$$\boxed{-\left[Z_m^{\perp}(\omega) \right]^* = Z_m^{\perp}(-\omega)}$$

  - As a consequence of the Panofsky-Wenzel theorem

$$\boxed{Z_m^{//}(\omega) = k \, Z_m^{\perp}(\omega)}$$

IMPEDANCES (3/8)

- **What is the coherent part of the transverse SC impedance (considering both electric and ac magnetic images)?**
  - **In the "SC course", we saw that the coherent horizontal force in a circular beam pipe is**

$$F_x^{SC,coh} = \frac{\lambda\, e}{2\,\pi\,\varepsilon_0\,\gamma^2}\,\frac{\overline{x}}{b^2} \quad \text{for } \overline{x} << \text{b}$$

$$\lambda = \frac{Q}{l} = \frac{N_b\, e}{l} \xrightarrow[l \to 0]{} Q\,\delta\left(s - \upsilon\, t\right)$$

$$z = s - \upsilon\, t$$

=> $$F_x^{SC,coh}\left(z\,;t\right) = \frac{e}{2\,\pi\,\varepsilon_0\,\gamma^2\,b^2}\,\delta\left(z\right) \times Q\,\overline{x}$$

$= Q_1$, with $a = \overline{x}$

$$W_1^{SC}\left(z\right) = -\frac{1}{q\,Q_1}\int_0^L F_x^{SC,coh}\,ds = -\frac{L\,F_x^{SC,coh}}{q\,Q_1} = -\frac{L}{2\,\pi\,\varepsilon_0\,\gamma^2\,b^2}\,\delta\left(z\right)$$

Behind the bunch

or $$W_1^{SC}\left(t\right) = -\frac{L}{2\,\pi\,\varepsilon_0\,\gamma^2\,b^2}\,\frac{\delta\left(t\right)}{\upsilon}$$

IMPEDANCES (4/8)

using the relation $$\delta(s-\upsilon t)=\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,e^{j\omega t}\left[\frac{e^{-jks}}{\upsilon}\right]$$

one has $$F_x^{SC,coh}(z;\omega)=\frac{e\,Q_1}{2\,\pi\,\varepsilon_0\,\gamma^2\,b^2}\,\frac{e^{-jks}}{\upsilon}$$

Fourier Transform (FT)

=> $$\mathrm{FT}\left[W_1^{SC}(t)\,e^{jks}\right]=-\frac{L}{2\,\pi\,\varepsilon_0\,\gamma^2\,b^2\,\upsilon}=-\frac{L\,Z_0}{2\,\pi\,\beta\,\gamma^2\,b^2}$$

Remembering that $Z_1^x(\omega)$ is the Fourier transform of $W_1(t)\,e^{jks}$ (with a - *j* added for the transverse plane) one finally obtains

=> $$\boxed{Z_1^{x,SC,coh}(\omega)=j\,\frac{L\,Z_0}{2\,\pi\,\beta\,\gamma^2\,b^2}}$$

IMPEDANCES (5/8)

- Another (equivalent, i.e. giving the same result) way to define the transverse impedance is often used and is given by

In time domain

$$\boxed{Z_1^x(\omega) = \frac{j}{Q_1} \int_0^L ds\, \mathrm{FT}\left(\frac{F_x}{q}\right) e^{\,j k s}}$$

- Finally, another (equivalent, i.e. giving the same result) way to define the impedance is => For coasting beams ($\lambda = \text{constant}$)

$$\boxed{Z_1^x(\omega) = \frac{j}{P_x} \int_0^L \mathrm{FT}\left(\frac{F_x}{q}\right) ds}$$

$P_x = I_b\, \overline{x}$

$I_b = \lambda\, \upsilon$

A *β* is also sometimes added in the denominator
to cancel the velocity effect in the Lorentz force (magnetic part)

IMPEDANCES (6/8)

- **What is the longitudinal SC impedance?**
  - **In the "SC course", we saw that the longitudinal space charge force for a uniform bunch in a circular beam pipe is**

$$F_s^{SC} = -\frac{e}{4\pi\varepsilon_0\gamma^2}\frac{d\lambda(z)}{dz}\left[1+2\ln\left(\frac{b}{a}\right)\right]$$

$$g_0 = 1+2\ln\left(\frac{b}{a}\right)$$

Depends on the source (it is 0 for a δ-function considered here)

Furthermore, $\lambda = Q\,\delta(z)$ **=>** $\frac{d\lambda(z)}{dz} = Q\,\delta'(z)$

=> $$F_s^{SC} = -\frac{e\,Q}{2\pi\varepsilon_0\gamma^2}\ln\left(\frac{b}{a}\right)\delta'(z)$$

$$W'_{0,SC}(z) = -\frac{1}{e\,Q}\int_0^L F_s^{SC}\,ds = \frac{L}{2\pi\varepsilon_0\gamma^2}\ln\left(\frac{b}{a}\right)\delta'(z)$$
$$= \frac{L\,Z_0\,c}{2\pi\gamma^2}\ln\left(\frac{b}{a}\right)\delta'(z) = \frac{L\,Z_0}{2\pi c\,\beta^2\gamma^2}\ln\left(\frac{b}{a}\right)\delta'(t)$$

$$Z_0^{//,SC}(\omega) = -j\frac{L\,\omega\,Z_0}{2\pi c\,\beta^2\gamma^2}\ln\left(\frac{b}{a}\right)$$

$$\delta'(z) = \frac{\delta'(t)}{\upsilon^2}$$

$$\mathrm{FT}\left[\delta'(t)\right] = -j\,\omega$$

IMPEDANCES (7/8)

- **More general definition of the impedances (still for a cylindrically symmetric structure)**

$$Z_m^{//}(\omega) = -\frac{1}{Q_m^2} \int dV \, E_m^{//} \, J_m^*$$

$$dV = r \, dr \, d\vartheta \, ds$$

$$Z_m^{\perp}(\omega) = -\frac{1}{k \, Q_m^2} \int dV \, E_m^{//} \, J_m^*$$

=> For the previous ring-shaped source, it yields

In frequency domain

$$Z_0^{//}(\omega) = -\frac{1}{Q_0} \int_0^L ds \, E_s(r = a) \, e^{j k s}$$

In frequency domain

$$Z_1^{\perp}(\omega) = -\frac{L}{k \pi a Q_1} \int_0^{2\pi} d\vartheta \, E_s(r = a, \vartheta, s) \cos\vartheta \; e^{j k s}$$

IMPEDANCES (8/8)

- As the conductivity, permittivity and permeability of a material depend in general on frequency, it is usually better (or easier) to treat the problem in the frequency domain (for a circular machine), i.e. compute the impedance
- It is also easier to treat the case $\boxed{\beta \neq 1}$
- Then, a Fourier transform is applied to obtain the wake field in the time domain
- General properties of impedances or wake fields
  - We already saw some of them before but there are more
  - Another one: Directional symmetry of impedance (Lorentz reciprocity theorem) => Same impedance from both sides if the entrance and exit are the same

GENERALIZED NOTION OF IMPEDANCE (1/24)

- **Axi-symmetric structures => A current density with some azimuthal Fourier component creates electromagnetic fields with the same azimuthal Fourier component**

$$\overline{Z}_m(\omega) = -\frac{1}{Q^2}\int dV\, \overline{E}_m\, \overline{J}_m^*$$

"Usual" definition of the longitudinal impedance (m=0,1,2,…) => In fact *Q* is used here instead of Q_m

with

$$\overline{J}_m = \frac{Q}{\pi a^{m+1}\left(1+\delta_{m0}\right)}\,\delta(r-a)\,\cos(m\vartheta)\,e^{-jks}$$

where $\overline{E}_m$ is the longitudinal electric field created by this current density

GENERALIZED NOTION OF IMPEDANCE (2/24)

- **Non axi-symmetric structures => A current density with some azimuthal Fourier component may create an electromagnetic field with various different azimuthal Fourier components => A more general beam coupling impedance is defined in order to treat coupling of different azimuthal Fourier components**

$$\boxed{Z_{m,n}(\omega) = -\frac{1}{Q^2} \int dV \, E_m \, J_n^*}$$

More "general" definition of the longitudinal impedance (m,n = 0, ±1, ±2,…)

with $$\boxed{J_n = \frac{Q}{2 \pi a^{|n|+1}} \delta(r-a) \, e^{j n \vartheta} \, e^{-j k s}}$$

where $\boxed{E_n}$ **is the longitudinal electric field created by this current density**

GENERALIZED NOTION OF IMPEDANCE (3/24)

- **Therefore,**

$$\overline{J}_0 = J_0$$

$$\overline{J}_m = J_m + J_{-m}$$

For m ≥ 1

and (assuming the principle of superposition)

For m ≥ 1

$$\overline{Z}_m(\omega) = -\frac{1}{Q^2}\int dV\left(E_m + E_{-m}\right)\left(J_m^* + J_{-m}^*\right)$$

=>

$$\overline{Z}_0 = Z_{0,0}$$

$$\overline{Z}_x \equiv \overline{Z}_1 = Z_{1,1} + Z_{1,-1} + Z_{-1,1} + Z_{-1,-1}$$

$$\overline{Z}_y \equiv \overline{Z}_1\left(\text{with}\cos\rightarrow\sin\right) = Z_{1,1} - Z_{1,-1} - Z_{-1,1} + Z_{-1,-1}$$

$$\overline{Z}_m = Z_{m,m} + Z_{m,-m} + Z_{-m,m} + Z_{-m,-m}$$

For m ≥ 1

GENERALIZED NOTION OF IMPEDANCE (4/24)

- Consider the case of a source particle at

$$\left| \begin{array}{l} x_1 = a_1 \cos \vartheta_1 \\ y_1 = a_1 \sin \vartheta_1 \end{array} \right.$$

and a test particle at

$$\left| \begin{array}{l} x_2 = a_2 \cos \vartheta_2 \\ y_2 = a_2 \sin \vartheta_2 \end{array} \right.$$

- The source current density (at the source particle) is given by

$$J_z = Q \, \delta(x - x_1) \, \delta(y - y_1) \, e^{-jks}$$

and

$$\delta(x - x_1) \, \delta(y - y_1) = \frac{1}{a_1} \delta(r - a_1) \, \delta_p(\vartheta - \vartheta_1)$$

$$= \frac{1}{a_1} \delta(r - a_1) \times \frac{1}{2\pi} \sum_{m=-\infty}^{m=+\infty} e^{jm(\vartheta - \vartheta_1)}$$

GENERALIZED NOTION OF IMPEDANCE (5/24)

=>

$$J_z = \frac{Q}{2\pi a_1}\,\delta(r-a_1)\,e^{-jks}\sum_{m=-\infty}^{m=+\infty} e^{jm(\vartheta-\vartheta_1)}$$

=>

$$J_z = Q\,\delta(x-x_1)\,\delta(y-y_1)\,e^{-jks} = \sum_{m=-\infty}^{m=+\infty} a_1^{|m|}\,e^{-jm\vartheta_1}\,J_m$$

- **The longitudinal impedance is given by**

$$Z = -\frac{1}{Q^2}\int dV\left(\sum_{m=-\infty}^{m=+\infty} a_1^{|m|}\,e^{-jm\vartheta_1}\,E_m\right)\left(\sum_{n=-\infty}^{n=+\infty} a_2^{|n|}\,e^{jn\vartheta_2}\,J_n^{*}\right) = \sum_{m,n} a_1^{|m|}\,a_2^{|n|}\,e^{-jm\vartheta_1}\,e^{jn\vartheta_2}\,Z_{m,n}$$

Electric field created by the source in (1)

Complex conjugate of the current density of the test particle in (2)

GENERALIZED NOTION OF IMPEDANCE (6/24)

which yields, up to the 2nd order,

$$\boxed{\begin{aligned}Z &= Z_{0,0} + (x_1 - j\,y_1)\,Z_{1,0} + (x_1 + j\,y_1)\,Z_{-1,0} + (x_2 + j\,y_2)\,Z_{0,1}\\ &+ (x_2 - j\,y_2)\,Z_{0,-1} + (x_1 - j\,y_1)^2\,Z_{2,0} + (x_1 - j\,y_1)(x_2 - j\,y_2)\,Z_{1,-1}\\ &+ (x_2 - j\,y_2)^2\,Z_{0,-2} + (x_1 - j\,y_1)(x_2 + j\,y_2)\,Z_{1,1} + (x_1 + j\,y_1)(x_2 - j\,y_2)\,Z_{-1,-1}\\ &+ (x_1 + j\,y_1)^2\,Z_{-2,0} + (x_1 + j\,y_1)(x_2 + j\,y_2)\,Z_{-1,1} + (x_2 + j\,y_2)^2\,Z_{0,2}\end{aligned}}$$

- **Applying Panofksy-Wenzel theorem (remembering that the transverse impedance is defined with an additional *j*)**

$$\boxed{k\,Z^{\perp} = \nabla_2^{\perp}\,Z}$$

=> $$\boxed{k\,Z_x = \frac{\partial Z}{\partial x_2}}$$ **and** $$\boxed{k\,Z_y = \frac{\partial Z}{\partial y_2}}$$

GENERALIZED NOTION OF IMPEDANCE (7/24)

- **The "general" transverse impedances $Z_{x,y}$ (not normalized by the transverse displacement) on a test particle at ($x_2 = a_2 \cos\theta_2$, $y_2 = a_2 \sin\theta_2$) from a source at ($x_1 = a_1 \cos\theta_1$, $y_1 = a_1 \sin\theta_1$), are thus given by (to 1st order)**

$$\begin{aligned} k\, Z_x = & \left(Z_{0,1} + Z_{0,-1} \right) + x_1 \, \overline{Z}_x + j\, y_1 \left(- Z_{1,-1} - Z_{1,1} + Z_{-1,-1} + Z_{-1,1} \right) \\ & + 2 \left(Z_{0,2} + Z_{0,-2} \right) x_2 + 2 \left(Z_{0,2} - Z_{0,-2} \right) j\, y_2 \end{aligned}$$

$$\begin{aligned} k\, Z_y = & \; j \left(Z_{0,1} - Z_{0,-1} \right) + y_1 \, \overline{Z}_y + j\, x_1 \left(- Z_{1,-1} + Z_{1,1} - Z_{-1,-1} + Z_{-1,1} \right) \\ & - 2 \left(Z_{0,2} + Z_{0,-2} \right) y_2 + 2 \left(Z_{0,2} - Z_{0,-2} \right) j\, x_2 \end{aligned}$$

=> $Z_x^{\text{driving}} = \overline{Z}_x \,/\, k$ $\quad$ $Z_y^{\text{driving}} = \overline{Z}_y \,/\, k$ $\quad$ $Z^{\text{detuning}} = - 2 \left(Z_{0,2} + Z_{0,-2} \right) / \, k$

GENERALIZED NOTION OF IMPEDANCE (8/24)

- **2-wire measurements** => Here , the current density by 2 wires at $x = \pm a$ is approximated by

$$\boxed{J = Q\left[\delta(x-a) - \delta(x+a)\right]\delta(y)\, e^{-jks}}$$

=> $$J = Q\left[\delta(x-a)\,\delta(y) - \delta(x+a)\,\delta(y)\right] e^{-jks}$$

$$= \frac{\delta(r-a)\,\delta_p(0)}{a} \qquad = \frac{\delta(r-a)\,\delta_p(\pi)}{a}$$

=> $$J = \frac{Q}{a}\,\delta(r-a)\left[\,\delta_p(0) - \delta_p(\pi)\right] e^{-jks}$$

GENERALIZED NOTION OF IMPEDANCE (9/24)

=>
$$J = \frac{Q}{2 \pi a} \delta (r - a) e^{- j k s} \left[\sum_{m = - \infty}^{m = + \infty} e^{j m \vartheta} - \sum_{m = - \infty}^{m = + \infty} e^{j m \vartheta} e^{- j m \pi} \right]$$

$$= \left| \begin{array}{l} 1 \text{ if } m \text{ is even} \\ -1 \text{ if } m \text{ is odd} \end{array} \right.$$

=>
$$J = \frac{Q}{\pi a} \delta (r - a) e^{- j k s} \sum_{m = - \infty}^{m = + \infty} e^{j (2 m + 1) \vartheta}$$

=>
$$J = 2 \sum_{m = - \infty}^{m = + \infty} a^{| 2 m + 1 |} J_{2 m + 1}$$

GENERALIZED NOTION OF IMPEDANCE (10/24)

=>

$$Z = -\frac{1}{Q^2}\int dV \left(2 \sum_{m=-\infty}^{m=+\infty} a^{\left|2m+1\right|} \, E_{2m+1} \right) \left(2 \sum_{n=-\infty}^{n=+\infty} a^{\left|2n+1\right|} \, J^*_{2n+1} \right)$$

$$= 4 \sum_{m,n} a^{\left|2m+1\right|} \, a^{\left|2n+1\right|} \, Z_{2m+1,2n+1}$$

$$= 4 \left(a^2 \, Z_{1,1} + a^2 \, Z_{-1,1} + a^2 \, Z_{1,-1} + a^2 \, Z_{-1,-1} \right)$$

$$= \left(2 \, a \right)^2 \, \overline{Z}_x$$

Up to 2nd order

=>

$$\boxed{Z_x^{\text{driving}} = \frac{\overline{Z}_x}{k} = \frac{\upsilon \, Z}{\omega \left(2 \, a \right)^2}}$$

=> If the longitudinal impedance Z can be measured (simulated), then the transverse (driving or dipolar) impedance can be deduced from 2-wire measurements (simulations)

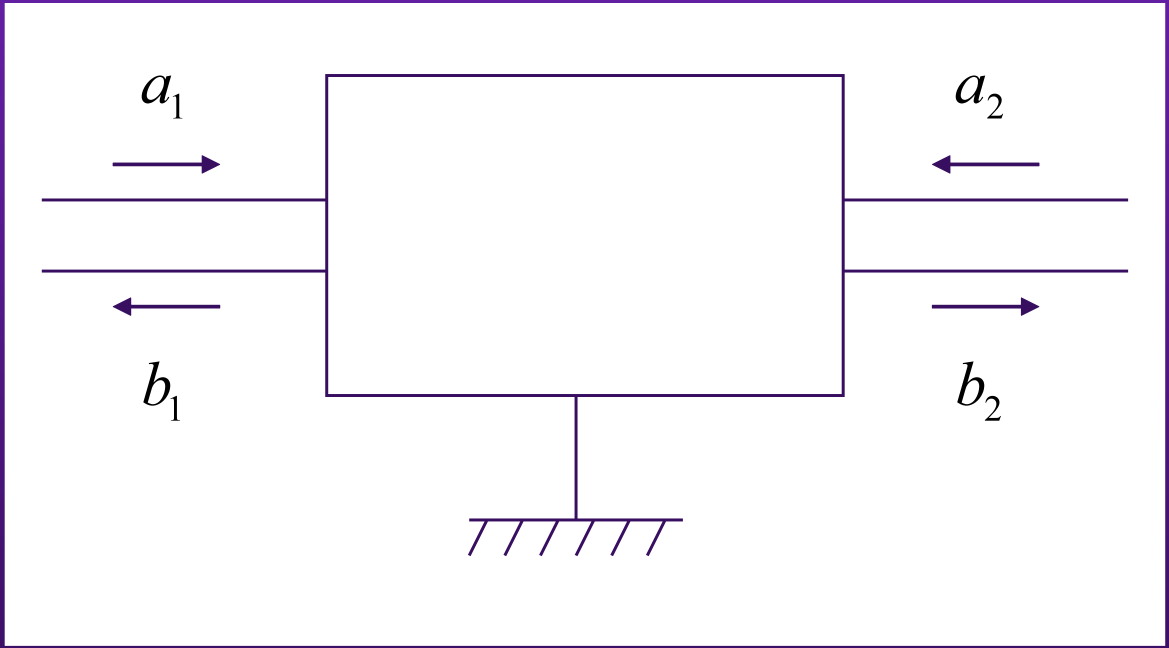
GENERALIZED NOTION OF IMPEDANCE (11/24)

Usually the longitudinal impedance $\boxed{Z}$ is calculated from the characteristic impedance $\boxed{Z_{ch}}$ and the scattering parameter $\boxed{S_{21}}$ as

$$Z = 2\, Z_{ch} \, \frac{1 - S_{21}}{S_{21}}$$

Characteristic impedance

Scattering parameter



Scattering matrix

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Direct (forward) wave

Reflected (backward) wave

GENERALIZED NOTION OF IMPEDANCE (12/24)

- 1-wire measurements => Here , the current density is approximated by

$$\boxed{J = Q\,\delta(x - x_0)\,\delta(y - y_0)\,e^{-jks}}$$

=> (see previous slides)

$$J = \sum_{m=-\infty}^{m=+\infty} a^{|m|}\, e^{-jm\vartheta_0}\, J_m$$

$$\left|\begin{array}{l} x_0 = a\cos\vartheta_0 \\ y_0 = a\sin\vartheta_0 \end{array}\right.$$

=>

$$Z = -\frac{1}{Q^2}\int dV \left(\sum_{m=-\infty}^{m=+\infty} a^{|m|}\, e^{-jm\vartheta_0}\, E_m\right)\left(\sum_{n=-\infty}^{n=+\infty} a^{|n|}\, e^{jn\vartheta_0}\, J_n^*\right)$$

$$= \sum_{m,n} a^{|m|+|n|}\, e^{-j(m-n)\vartheta_0}\, Z_{m,n}$$

GENERALIZED NOTION OF IMPEDANCE (13/24)

=> $$Z = A_1 + a\,e^{-j\vartheta_0}\,A_2 + a\,e^{j\vartheta_0}\,A_3 + a^2\,e^{-2j\vartheta_0}\,A_4 + a^2\,e^{2j\vartheta_0}\,A_5 + a^2\,A_6$$

Up to 2nd order

with

$$A_1 = Z_{0,0}$$

$$A_2 = Z_{1,0} + Z_{0,-1}$$

$$A_3 = Z_{0,1} + Z_{-1,0}$$

$$A_4 = Z_{2,0} + Z_{1,-1} + Z_{0,-2}$$

$$A_5 = Z_{0,2} + Z_{-1,1} + Z_{-2,0}$$

$$A_6 = Z_{1,1} + Z_{-1,-1}$$

GENERALIZED NOTION OF IMPEDANCE (14/24)

- **If there is top/bottom and left/right symmetry (fortunately it is the usual case…), the situation simplifies a lot**
  - **If a = x_0 and $\theta_0 = 0$**

$$Z = A_1 + x_0^2 \left(A_4 + A_5 + A_6 \right)$$
$$= A_1 + x_0^2 \left[\overline{Z}_x + \left(Z_{2,0} + Z_{0,2} + Z_{-2,0} + Z_{0,-2} \right) \right]$$

Scanning x_0 gives a parabola

  - **If a = y_0 and $\theta_0 = \pi / 2$**

$$Z = A_1 + y_0^2 \left(- A_4 - A_5 + A_6 \right)$$
$$= A_1 + y_0^2 \left[\overline{Z}_y - \left(Z_{2,0} + Z_{0,2} + Z_{-2,0} + Z_{0,-2} \right) \right]$$

GENERALIZED NOTION OF IMPEDANCE (15/24)

=> IF $\boxed{Z_{2,0} + Z_{-2,0} = Z_{0,2} + Z_{0,-2}}$ **, then**

Still has to be demonstrated in the general case

$$Z_{2,0} + Z_{0,2} + Z_{-2,0} + Z_{0,-2} = 2 \left(Z_{0,2} + Z_{0,-2} \right) = - k \, Z^{\text{detuning}}$$

=>

$$\boxed{Z = A_1 + k \, x_0^2 \left[Z_x^{\text{driving}} - Z_x^{\text{detuning}} \right]}$$

$$\boxed{Z = A_1 + k \, y_0^2 \left[Z_y^{\text{driving}} + Z_y^{\text{detuning}} \right]}$$

Therefore, with 1-wire measurements, only the difference in x and sum in y of the driving and detuning impedances can be obtained

GENERALIZED NOTION OF IMPEDANCE (16/24)

- **If there is NO top/bottom or left/right symmetry, the situation is more involved:**
  - By scanning a and θ_0 (i.e. measuring Z for different values of a and θ_0), $A_{1,2,3,4,5,6}$ can be found
  - Then, using the 2-wire technique the dipolar (driving) impedances can be obtained: $Z_x^{\text{driving}} = \overline{Z}_x / k$ $Z_y^{\text{driving}} = \overline{Z}_y / k$
  - Then compute $Z_{1,-1} + Z_{-1,1} = \left(\overline{Z}_x - \overline{Z}_y \right) / 2$
  - Then, if $Z_{2,0} + Z_{-2,0} = Z_{0,2} + Z_{0,-2}$,

$$\boxed{Z^{\text{detuning}} = \frac{Z_x^{\text{driving}} - Z_y^{\text{driving}}}{2} - \frac{A_4 + A_5}{k}}$$

GENERALIZED NOTION OF IMPEDANCE (17/24)

- Both 1-wire and 2-wire techniques are required (in asymmetric structures) to obtain all the information needed to correctly understand/describe the collective effects in accelerators

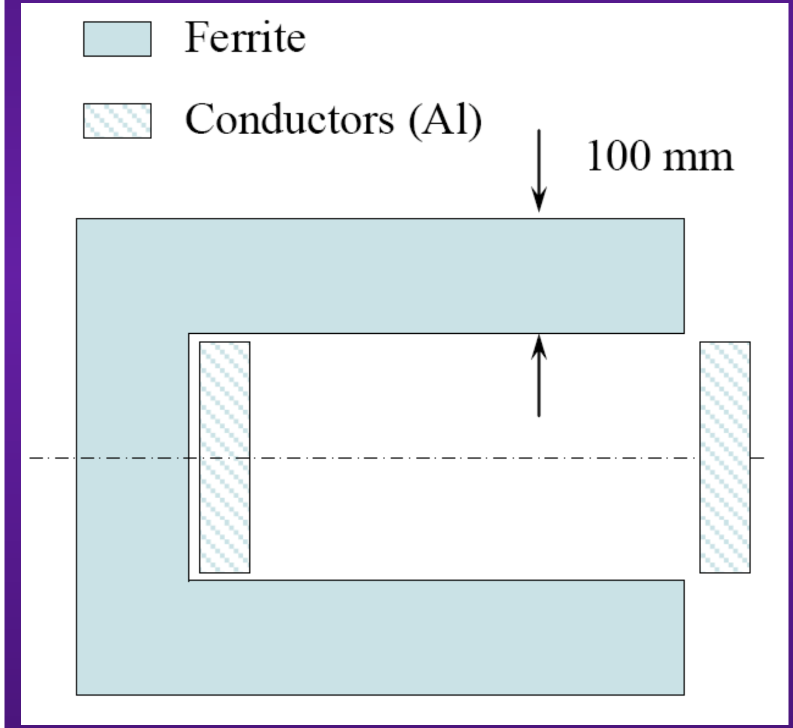
- With 2 wires the transverse dipolar (driving) impedances are obtained

- With 1 wire (scanning the wire position), and using the driving impedances measured with 2 wires, the detuning impedance can be deduced (IF a certain condition is fulfilled => Still to be checked in which cases this relation is satisfied or not)

- The coupling (and high order) terms are (usually) neglected, but could also be important in some cases

GENERALIZED NOTION OF IMPEDANCE (18/24)

- **Example of impedance measurement with 1 wire => Kicker KFA13 in the CERN PS**



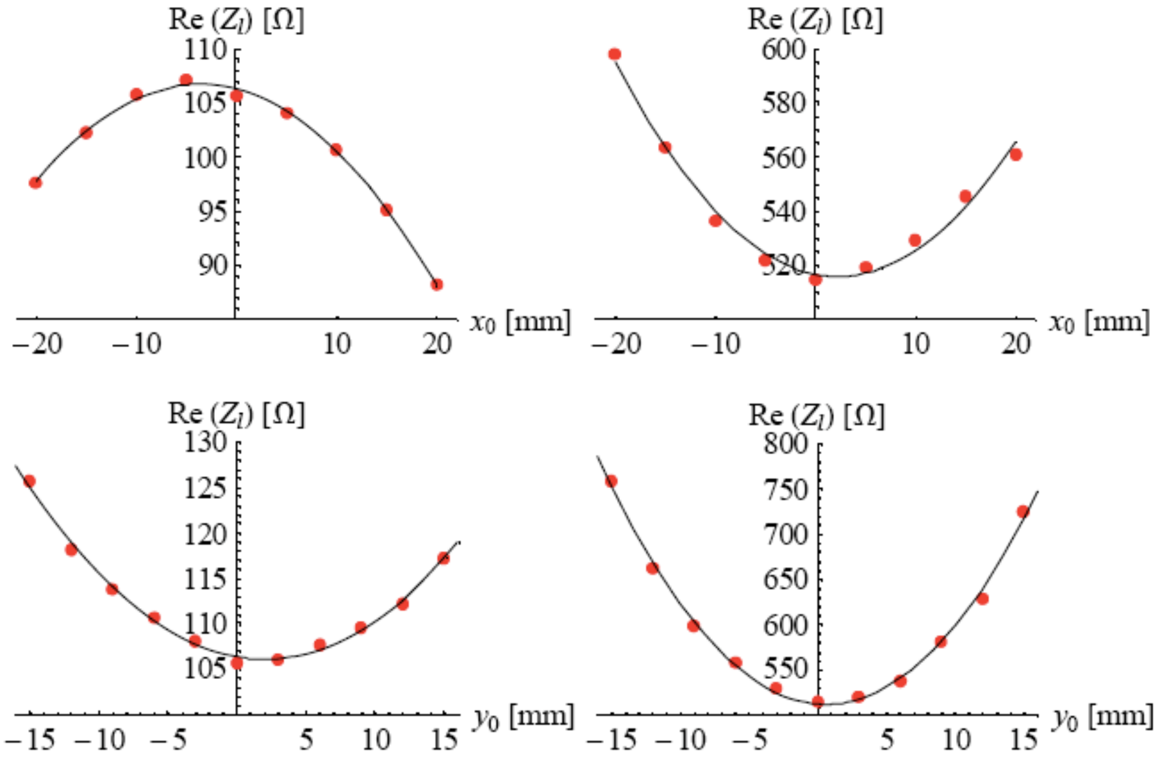
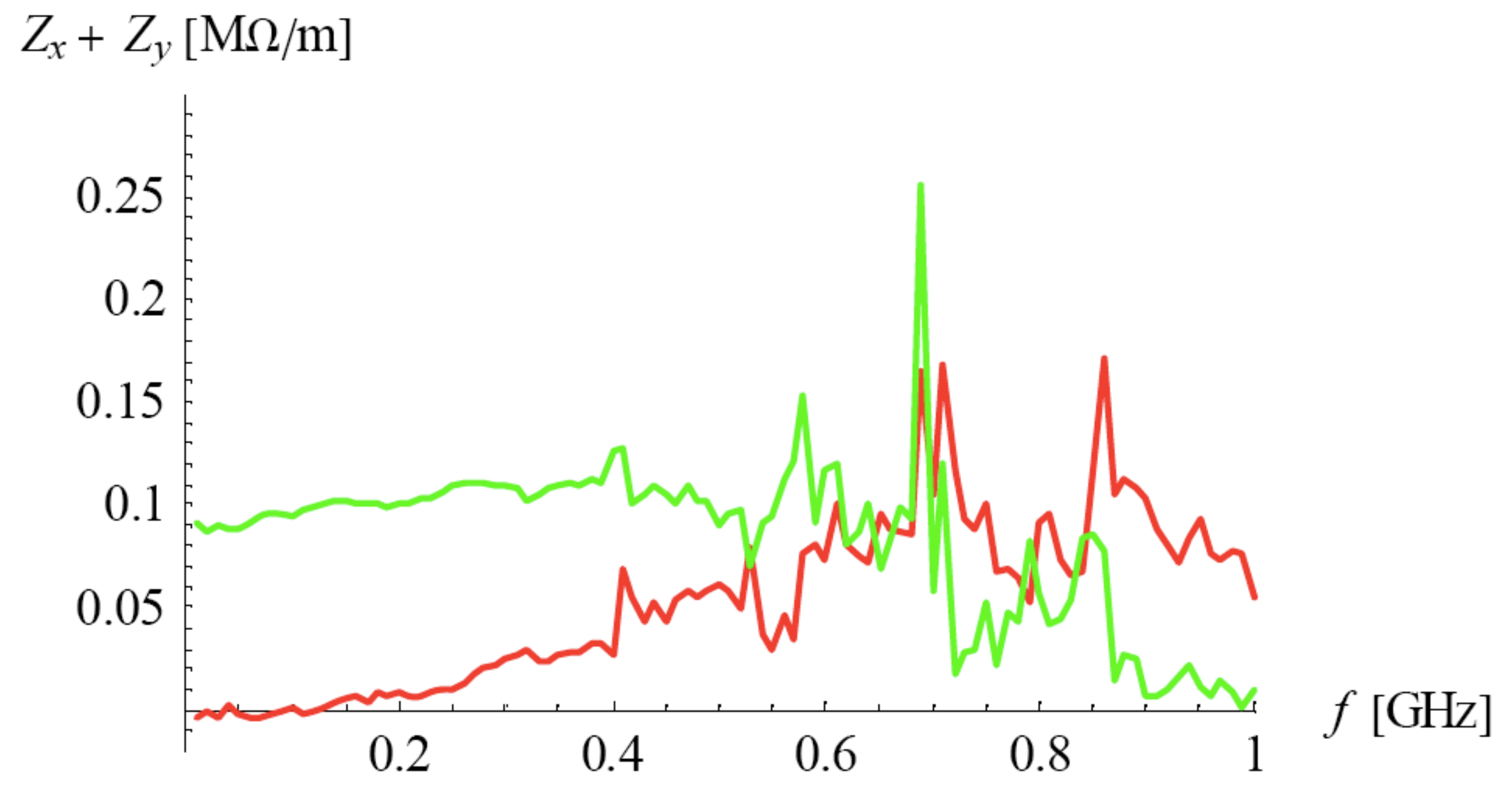


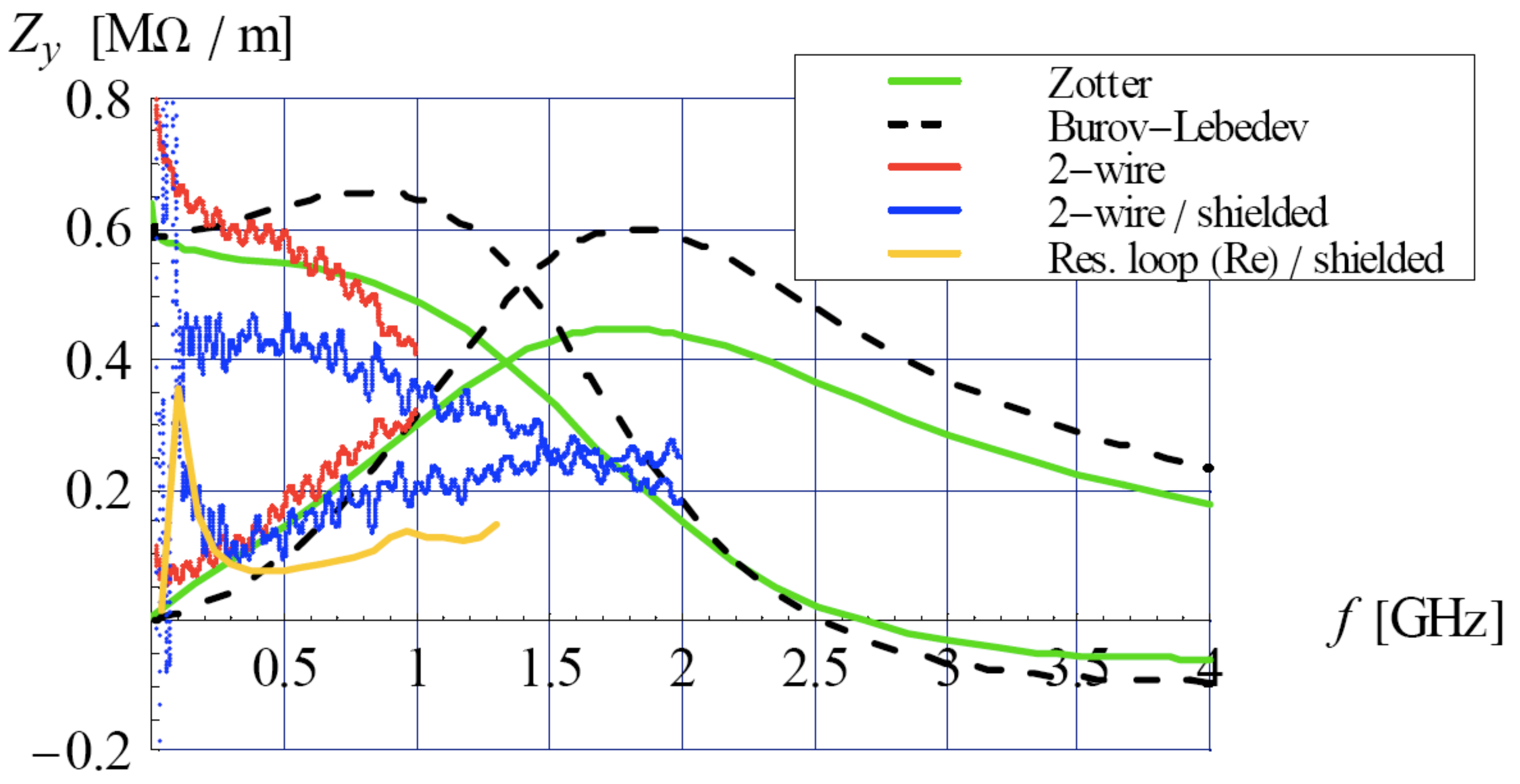
Figure 3: Measured real part of the longitudinal impedance (red dots) vs. (upper/lower) horizontal/vertical offset at 200 MHz (left) and 1 GHz (right). The full black line is the parabolic fit used to deduce the transverse impedance.

GENERALIZED NOTION OF IMPEDANCE (19/24)



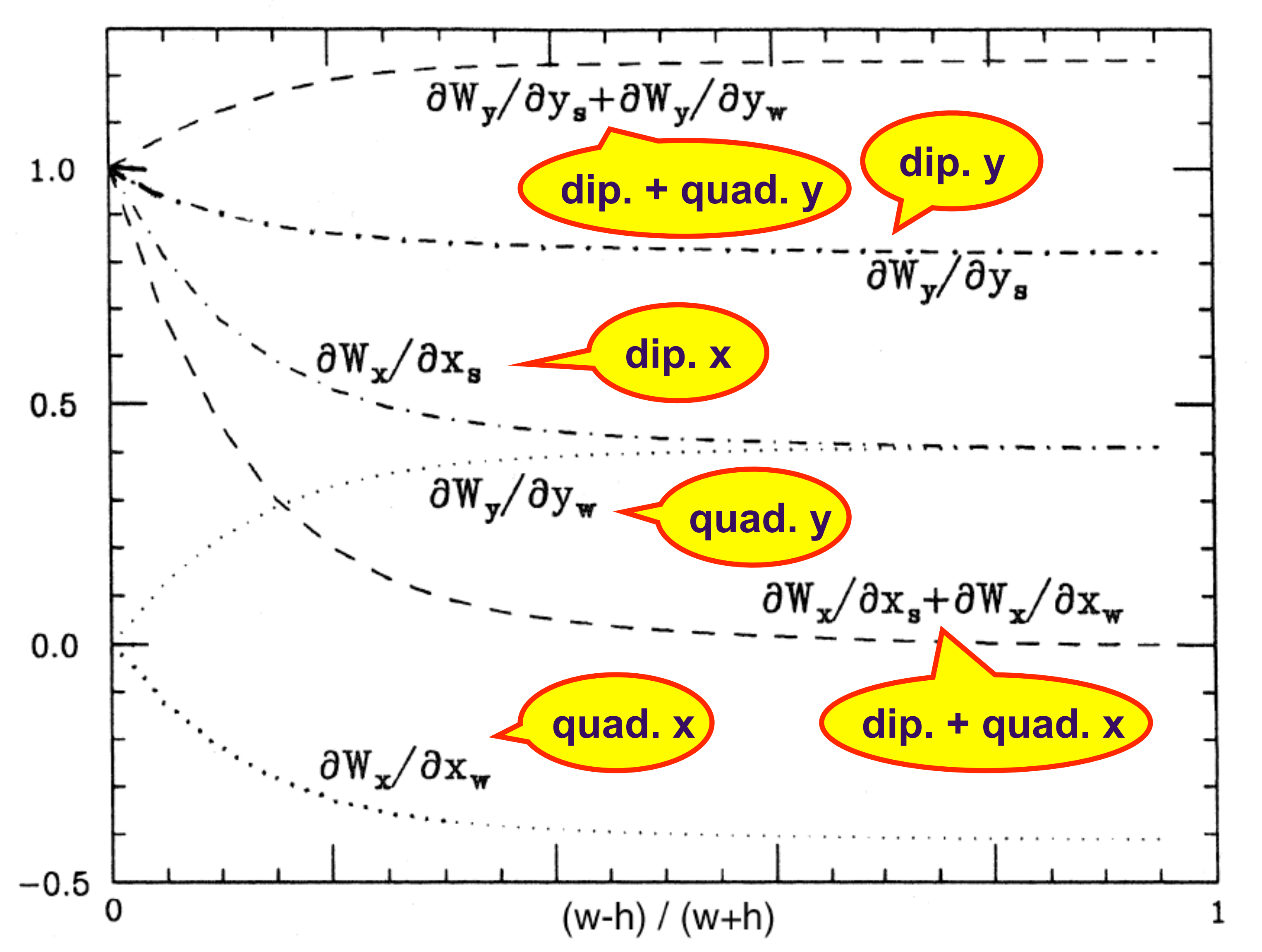
GENERALIZED NOTION OF IMPEDANCE (20/24)

- **Example of impedance measurement with 2 wires => A MKE kicker in the CERN SPS**



GENERALIZED NOTION OF IMPEDANCE (21/24)

- **Yokoya factors for dipolar and quadrupolar impedances in resistive elliptical pipes (compared to a circular one)**



$$\frac{\pi^2}{8}$$

$$\frac{\pi^2}{12}$$

$$\frac{\pi^2}{24}$$

$$-\frac{\pi^2}{24}$$

GENERALIZED NOTION OF IMPEDANCE (22/24)

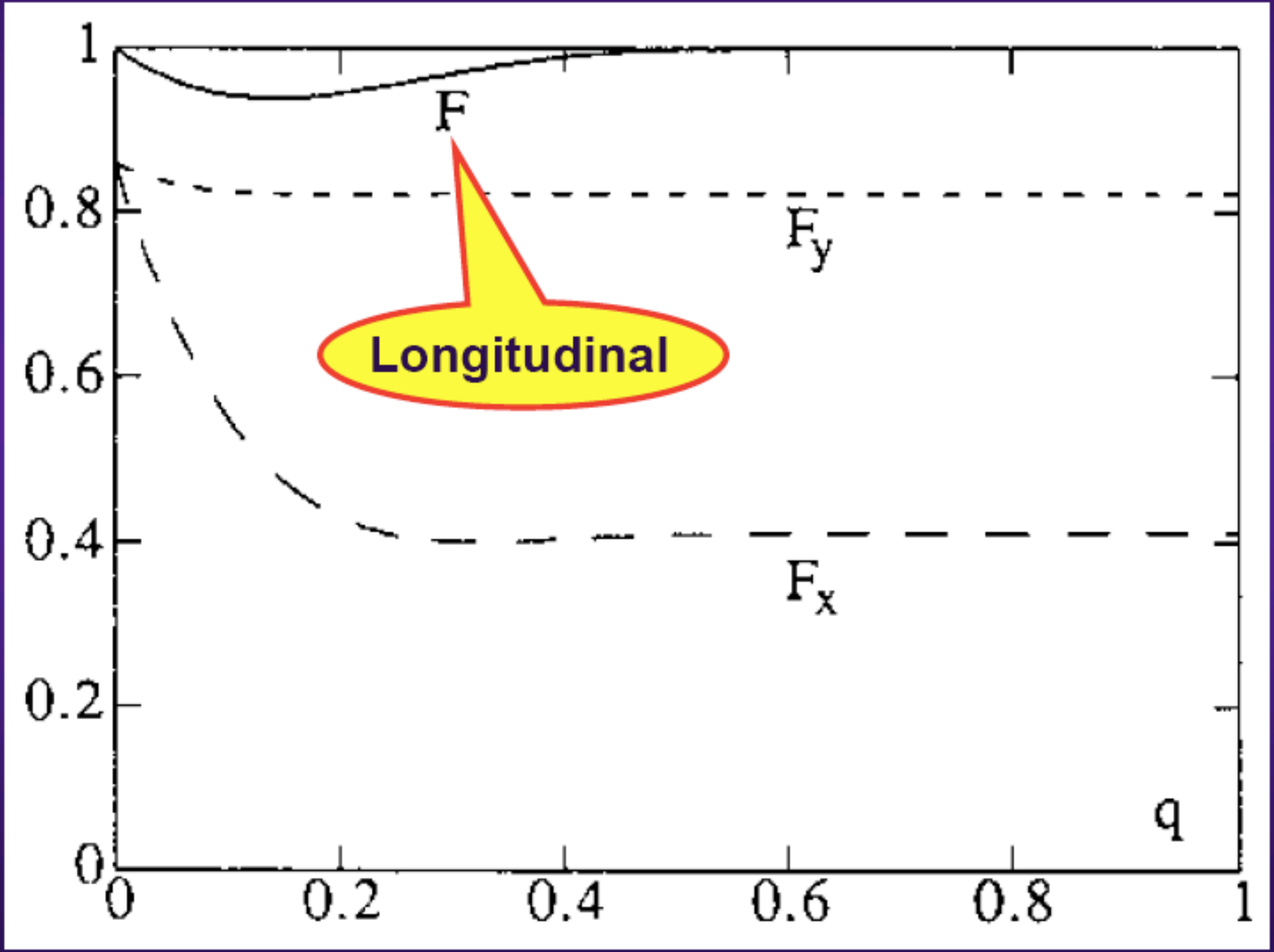
Form factors for a rectangular pipe :

$F(\lambda)$ **and** $F_{x,y}(\lambda)$ **with** $\lambda = \dfrac{b}{h}$

$$q = \frac{h-b}{h+b}$$

$h = \text{pipe half} - \text{width}$

$b = \text{pipe half} - \text{height}$



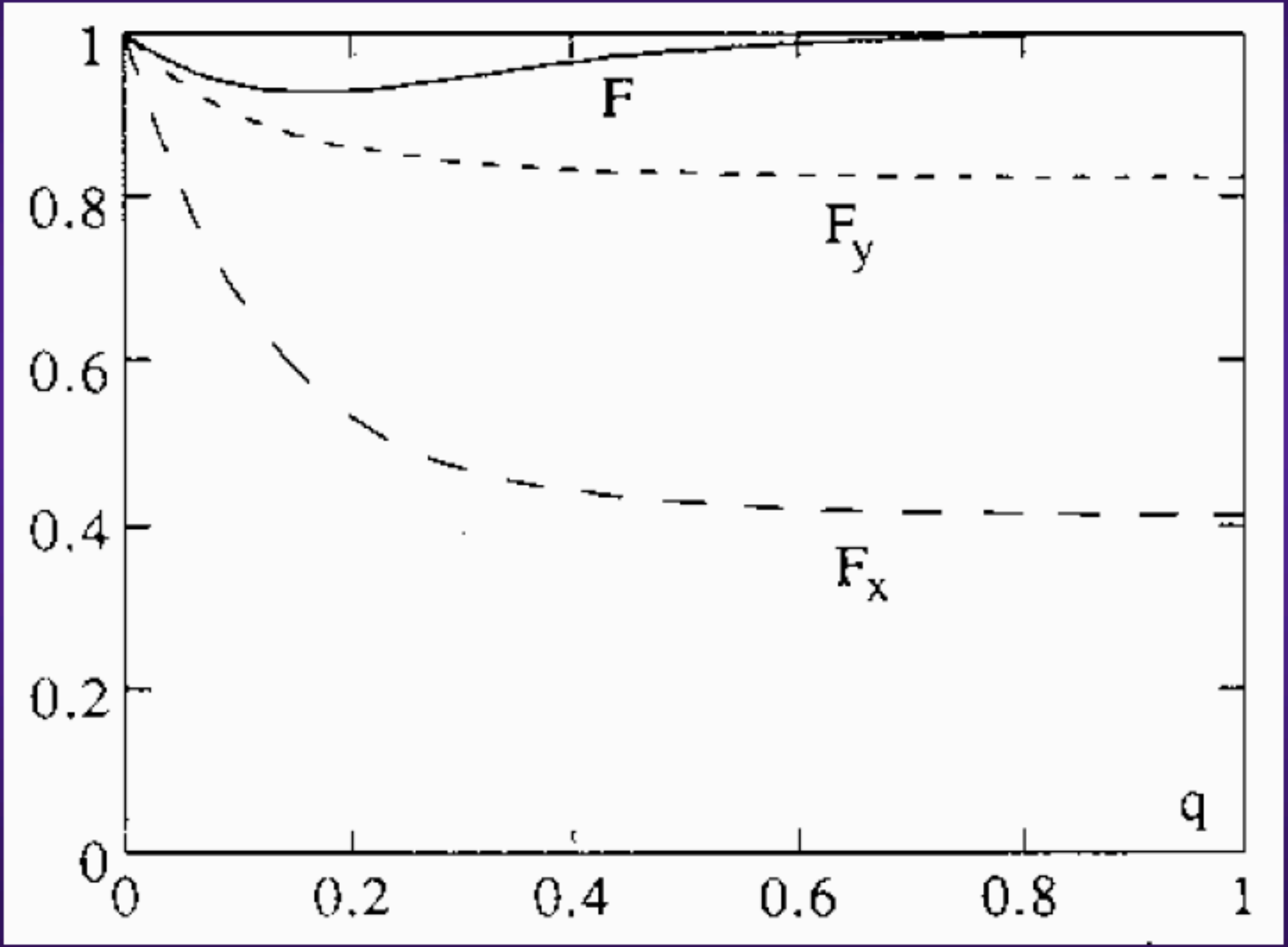
$$\frac{\pi^2}{12}$$

$$\frac{\pi^2}{24}$$

GENERALIZED NOTION OF IMPEDANCE (23/24)

Form factors for an elliptical pipe :

$F\left(u_0\right)$ **and** $F_{x,y}\left(u_0\right)$ **with** $q = e^{-2u_0}$



$\frac{\pi^2}{12}$

$\frac{\pi^2}{24}$

GENERALIZED NOTION OF IMPEDANCE (24/24)

- **Finally, the transverse impedances (dipolar and quadrupolar) should be weighted by the betatron function at the location of the impedance => This is what matters for the effect of a transverse impedance on the beam**

$$\boxed{\rightarrow \frac{\beta_x}{\beta_x^{average}} \times Z_\perp}$$

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (1/31)

◆ 1) Maxwell equations

- **In the frequency domain, time derivatives are replaced by** $j\omega$
- **Combining the conduction and displacement current terms yields**

$$curl\ \vec{H} = \rho\,\vec{\upsilon} + j\,\omega\,\varepsilon_c\ \vec{E}$$

$$div\ \vec{H} = 0$$

$$curl\ \vec{E} = -\ j\,\omega\,\mu\ \vec{H}$$

$$div\ \vec{E} = \frac{\rho}{\varepsilon_c}$$

with

$$\vec{B} = \mu\ \vec{H}$$

$$\mu = \mu_0\ \mu_1 = \mu_0\,\mu_r\left(1 - j\tan\vartheta_M\right)$$

$$\vec{D} = \varepsilon_c\ \vec{E}$$

$$\varepsilon_c = \varepsilon_0\ \varepsilon_1 = \varepsilon_0\left(\varepsilon_r' - j\ \varepsilon_r''\right) = \varepsilon_0\ \varepsilon_b + \frac{\sigma}{j\,2\pi\,f}$$

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (2/31)

- **2) Scalar Helmholtz equations for the longitudinal field components**

Using $curl\ curl = grad\ div - \Delta$, one obtains (using the circular cylindrical coordinates r, θ, s and assuming the source velocity to be along the s axis)

$$\boxed{\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial s^2}+\omega^2\mu\,\varepsilon_c\right]H_s=0}$$

$$\boxed{\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial s^2}+\omega^2\mu\,\varepsilon_c\right]E_s=\frac{1}{\varepsilon_c}\frac{\partial\,\rho}{\partial\,s}+j\,\omega\,\mu\,\rho\,\upsilon}$$

- **The homogeneous equation can be solved by separation of variables**

$$H_s \text{ or } E_s = \Theta(\theta)\,S(s)\,R(r)$$

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (3/31)

$\Rightarrow$ $\boxed{\Theta(\theta) = e^{\pm j m \theta}}$

m is called the azimuthal mode number (m=1 for pure dipole oscillations)

and $\boxed{S(s) = e^{\pm j k s}}$

k is called the wave number

The axial motion is seen to be a wave with phase velocity $\upsilon = \beta\, c = \omega / k$

which may in general differ from the beam velocity $\upsilon_b = \beta_b\, c$

R (r) is given by

$$\frac{1}{r}\frac{d}{d\,r}\left(r\,\frac{d\,R}{d\,r}\right) - \left(\frac{m^2}{r^2} + \nu^2\right) R = 0$$

$$\boxed{\nu = k\sqrt{1 - \beta^2\,\varepsilon_1\,\mu_1}}$$

Radial propagation constant

The solutions of this differential equation are the modified Bessel functions $I_m(\nu\, r)$ and $K_m(\nu\, r)$

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (4/31)

- 3) Source of the fields: Ring-beam distribution ⇒ Infinitesimally short, annular beam of charge $Q = N_b\, e$ and radius a traveling with velocity $\upsilon = \beta\, c$ along the s axis (equal to the bunch velocity)
  - Charge density in the frequency domain (see previous slides)

$$\rho(r,\vartheta,s;\omega) = \sum_{m=0}^{\infty} \frac{Q_m \cos(m\vartheta)}{\upsilon\,\pi\, a^{m+1}\left(1+\delta_{m0}\right)}\,\delta(r-a)\,e^{-jks}$$

=> $\rho_m \propto \cos(m\vartheta)\, e^{-jks}$

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (5/31)

- **Conclusion for the homogeneous scalar Helmholtz equations**
  - For **pure dipole oscillations excited by a horizontal cosine modulation** propagating along the particle beam, one can write the solutions for H_s and E_s as

$$\boxed{H_s = \sin(m\theta)\, e^{-jks} \left[C_1\, I_m(\nu r) + C_2\, K_m(\nu r) \right]}$$

$$\boxed{E_s = \cos(m\theta)\, e^{-jks} \left[C_3\, I_m(\nu r) + C_4\, K_m(\nu r) \right]}$$

$C_{1,2,3,4}$ are constants to be determined

  - Sine and cosine are interchanged for a purely vertical excitation (see source fields)
  - Only the solutions of the homogeneous Helmholtz equations are needed since all the regions considered are source free except the one containing the beam where the **source terms** have to be determined separately

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (6/31)

- 4) Transverse field components deduced from the longitudinal ones using Maxwell equations (in a source-free region)

$$\vec{G} = Z_0 \vec{H}$$

$$E_s = E_{s0} \cos(m\theta)$$

$$E_r = E_{r0} \cos(m\theta)$$

$$G_\theta = G_{\theta 0} \cos(m\theta)$$

$$G_s = G_{s0} \sin(m\theta)$$

$$G_r = G_{r0} \sin(m\theta)$$

$$E_\theta = E_{\theta 0} \sin(m\theta)$$

$$E_{r0} = \frac{j\,k}{\nu^2} \left(\beta\,\mu_1 \frac{m\,G_{s0}}{r} + \frac{d\,E_{s0}}{d\,r} \right)$$

$$E_{\theta 0} = -\frac{j\,k}{\nu^2} \left(\frac{m\,E_{s0}}{r} + \beta\,\mu_1 \frac{d\,G_{s0}}{d\,r} \right)$$

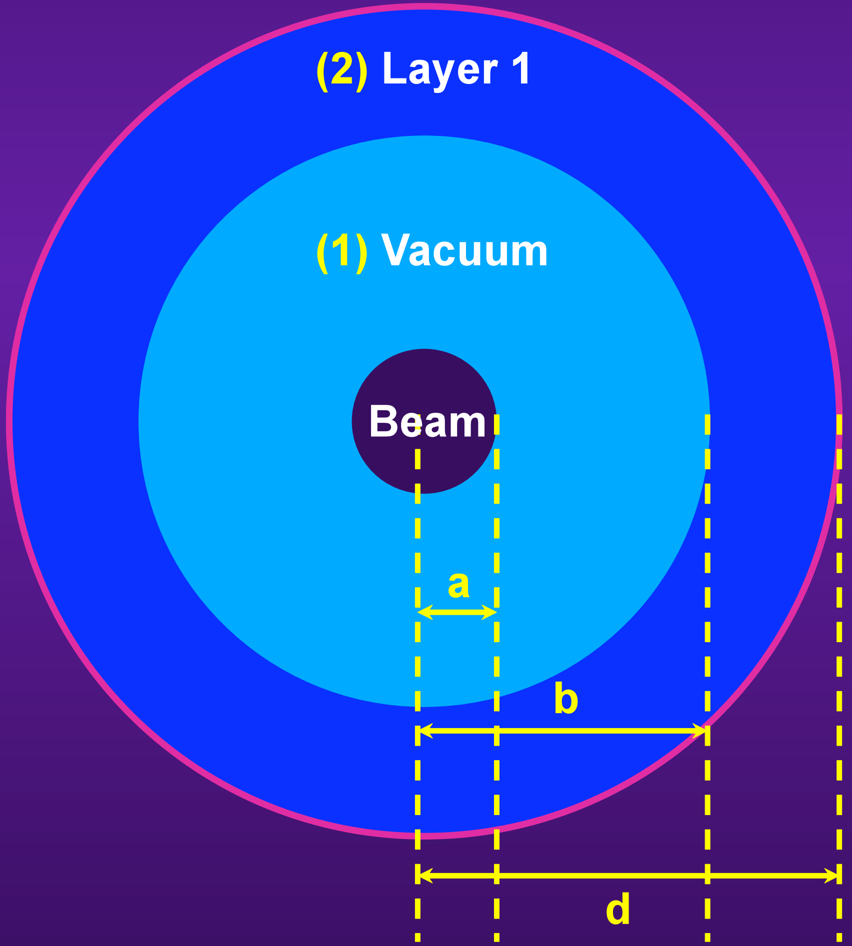
$$G_{r0} = \frac{j\,k}{\nu^2} \left(\beta\,\varepsilon_1 \frac{m\,E_{s0}}{r} + \frac{d\,G_{s0}}{d\,r} \right)$$

$$G_{\theta 0} = \frac{j\,k}{\nu^2} \left(\frac{m\,G_{s0}}{r} + \beta\,\varepsilon_1 \frac{d\,E_{s0}}{d\,r} \right)$$

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (7/31)

◆ 5) Let's consider the case of the transverse impedance (m = 1)

$$\rho_1 = \frac{Q_1 \cos(\vartheta)}{\upsilon\, \pi\, a^2}\, \delta(r-a)\, e^{-jks}$$



IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (8/31)

- Longitudinal source terms ⇒ Valid for $a \leq r \leq b$, i.e. in the vacuum between the beam and the pipe = region (1)

$$\boxed{E_s^{(s)}\left(r, \vartheta, s\right) = j\, C \cos\vartheta\; F_1\left(u\right)}$$

$$\boxed{G_s^{(s)}\left(r, \vartheta, s\right) = j\, C \sin\vartheta\; \alpha_{\text{TE}}\, I_1\left(u\right)}$$

with

$$C = \frac{\omega\, Q_1}{\pi\, a\, \varepsilon_0\, \upsilon^2\, \gamma^2}\, I_1\left(x_0\right) e^{-j\,k\,s} \qquad u = \frac{k\, r}{\gamma} \qquad x_0 = \frac{k\, a}{\gamma} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$F_1\left(u\right) = K_1\left(u\right) - \alpha_{\text{TM}}\, I_1\left(u\right)$$

α_{TE} and α_{TM} will be determined by the boundary conditions at b and d

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (9/31)

- The transverse components (in the same region) are then

$$E_\vartheta^{(s)}\left(r,\vartheta,s\right) = \gamma\, C \sin\vartheta \left[\frac{F_1\left(u\right)}{u} + \beta\, \alpha_{\mathrm{TE}}\, I_1'\left(u\right)\right]$$

$$G_\vartheta^{(s)}\left(r,\vartheta,s\right) = -\beta\, \gamma\, C \cos\vartheta \left[F_1'\left(u\right) + \frac{\alpha_{\mathrm{TE}}}{\beta}\, \frac{I_1\left(u\right)}{u}\right]$$

$$E_r^{(s)}\left(r,\vartheta,s\right) = -\gamma\, C \cos\vartheta \left[F_1'\left(u\right) + \beta\, \alpha_{\mathrm{TE}}\, \frac{I_1\left(u\right)}{u}\right]$$

$$G_r^{(s)}\left(r,\vartheta,s\right) = -\beta\, \gamma\, C \sin\vartheta \left[\frac{F_1\left(u\right)}{u} + \frac{\alpha_{\mathrm{TE}}}{\beta}\, I_1'\left(u\right)\right]$$

The quantity which enters in the transverse impedance is

$$\boxed{E_\vartheta^{(s)} + \upsilon\, B_r^{(s)} = E_\vartheta^{(s)} + \beta\, G_r^{(s)} = \frac{C \sin\vartheta}{\gamma} \times \frac{F_1\left(u\right)}{u}}$$

=> It depends only on α_{TM} and NOT on α_{TE} !

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (10/31)

- 6) Field matching
  - At the interfaces of 2 layers (r = constant) all field strength components have to be matched, i.e. in the absence of surface charges and currents the tangential field strengths $E_{s,\theta}$ and $H_{s,\theta}$ have to be continuous
  - Matching of the radial components is redundant
  - At a Perfect Conductor (PC) : $E_s = E_\theta = 0 \Rightarrow dG_s / dr = 0$
  - At a Perfect Magnet (PM) : $G_s = G_\theta = 0 \Rightarrow dE_s / dr = 0$
  - At r = Infinity $\Rightarrow$ Only $K_1(x)$ is permitted as $I_1(x)$ diverges

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (11/31)

- **7) The total (i.e. resistive-wall + space charge) horizontal impedance**

$$\begin{aligned} Z_x^{\text{Total}}(f) &= \frac{j}{Q_1}\int_{-\infty}^{+\infty} ds\left[E_x - \upsilon_b\, B_y\right] e^{jks} \\ &= \frac{j}{Q_1}\int_{-\infty}^{+\infty} ds\left[E_\vartheta^{(s)}\left(a, -\frac{\pi}{2}, s\right) + \upsilon_b\, B_r^{(s)}\left(a, -\frac{\pi}{2}, s\right)\right] e^{jks} \end{aligned}$$

$$\Rightarrow \quad Z_x^{\text{Total}}(f) = -\frac{j\, L\, Z_0\, I_1(x_0)\, K_1(x_0)}{\pi\, a^2\, \beta\, \gamma^2} + \alpha_{\text{TM}}\, \frac{j\, L\, Z_0\, I_1^2(x_0)}{\pi\, a^2\, \beta\, \gamma^2}$$

with *L* the length of the resistive pipe and $Z_0 = 120\,\pi$

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (12/31)

- **The “wall impedance” (and not the “resistive-wall impedance”) is obtained by subtracting from the total impedance, the “incoherent part” of the impedance (i.e. which does not depend on the wall, and comes from the direct space charge interaction) given by**

$$Z_x^{\mathrm{SC,incoh}}\left(f\right) = -\frac{j\,L\,Z_0\,I_1\left(x_0\right)K_1\left(x_0\right)}{\pi\,a^2\,\beta\,\gamma^2}$$

- **If** $x_0 << 1$ $\Rightarrow$ $I_1\left(x_0\right) \approx \frac{x_0}{2}$ **and** $K_1\left(x_0\right) \approx \frac{1}{x_0}$

$$\Rightarrow \quad \boxed{Z_x^{\mathrm{SC,incoh}}\left(f\right) = -\frac{j\,L\,Z_0}{2\,\pi\,a^2\,\beta\,\gamma^2} = -\frac{j\,L\,Z_0}{2\,\pi\,a^2\,\beta}\left(1-\beta^2\right)}$$

Electric **Magnetic**

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (13/31)

- **The present formalism can also be used for any number of layers of the vacuum pipe. The result for a single layer extending up to infinity is given below**

$$\boxed{\alpha_{\mathrm{TM}} = \frac{K_1(x_1)}{I_1(x_1)}\left[1 + \frac{\gamma\nu\,(P_1 - Q_1)\,(\beta\,x_1\,x_2)^2\,(\gamma\nu\,P_1 - k\,\mu_1\,Q_2)}{(\gamma\nu\,x_2 - k\,x_1)^2 - (\beta\,x_1\,x_2)^2\,(\gamma\nu\,P_1 - k\,\mu_1\,Q_2)\,(\gamma\nu\,P_1 - k\,\varepsilon_1\,Q_2)}\right]}$$

$$\boxed{\alpha_{\mathrm{TE}} = \frac{K_1(x_1)}{I_1(x_1)} \times \frac{\gamma\nu\beta\,x_1\,x_2\,(P_1 - Q_1)\,(\gamma\nu\,x_2 - k\,x_1)}{(\gamma\nu\,x_2 - k\,x_1)^2 - (\beta\,x_1\,x_2)^2\,(\gamma\nu\,P_1 - k\,\mu_1\,Q_2)\,(\gamma\nu\,P_1 - k\,\varepsilon_1\,Q_2)}}$$

with $\boxed{x_1 = \frac{k\,b}{\gamma}}$ $\boxed{x_2 = \nu\,b}$ $\boxed{P_1 = \frac{I_1'(x_1)}{I_1(x_1)}}$ $\boxed{Q_1 = \frac{K_1'(x_1)}{K_1(x_1)}}$ $\boxed{Q_2 = \frac{K_1'(x_2)}{K_1(x_2)}}$

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (14/31)

$$
\begin{aligned}
\Rightarrow \quad Z_x^{\mathrm{Wall,\,1\,layer}}(f) = {} & \frac{j\,L\,Z_0\,I_1^2(x_0)\,K_1(x_1)}{\pi\,a^2\,\beta\,\gamma^2\,I_1(x_1)} \\
& + j\,L\,Z_0\,\beta\,I_1^2(x_0)\,K_1(x_1)\,x_1^2\,x_2^2\,\gamma\,\nu \left(\frac{I_1'(x_1)}{I_1(x_1)} - \frac{K_1'(x_1)}{K_1(x_1)} \right) \\
& \times \left(\gamma\,\nu\,\frac{I_1'(x_1)}{I_1(x_1)} - k\,\mu_1\,\frac{K_1'(x_2)}{K_1(x_2)} \right) / \\
& \left\{ \begin{aligned} & \pi\,a^2\,\gamma^2\,I_1(x_1) \\ & \times \left[\begin{aligned} & (\gamma\,\nu\,x_2 - k\,x_1)^2 - (\beta\,x_1\,x_2)^2 \left(\gamma\,\nu\,\frac{I_1'(x_1)}{I_1(x_1)} - k\,\mu_1\,\frac{K_1'(x_2)}{K_1(x_2)} \right) \\ & \times \left(\gamma\,\nu\,\frac{I_1'(x_1)}{I_1(x_1)} - k\,\varepsilon_1\,\frac{K_1'(x_2)}{K_1(x_2)} \right) \end{aligned} \right] \end{aligned} \right\}
\end{aligned}
$$

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (15/31)

- In the case of 2 layers, the situation is more involved and the impedance is given by (for the single layer extending up to infinity)

$$\boxed{Z_x^{\text{Wall}\,,\,2\,\text{layers}}(f) = \frac{j\,L\,Z_0\,I_1^2(x_0)\,K_1(x_1)}{\pi\,a^2\,\beta\,\gamma^2\,I_1(x_1)} + \frac{j\,L\,Z_0\,I_1^2(s)}{\pi\,a^2\,\beta\,\gamma^2}\,\frac{K_1(x_1)}{I_1(x_1)}\,E_2\,(\alpha_2 - 1)}$$

where the parameters ($\boxed{E_2}$, $\boxed{\alpha_2}$) are 2 parameters out of 4 ($\boxed{\alpha_2}$, $\boxed{\eta_2}$, $\boxed{E_2}$ and $\boxed{G_2}$), solutions of the system of 4 linear equations

$$\gamma\,\nu_2\,x_2\,E_2(1-\alpha_2) + \gamma\,\nu_2\,x_1\,x_2\,\beta\,G_2(1-\eta_2)\,P_1 = k\,x_1\,E_2(1-\alpha_2) + k\,x_1\,x_2\,\beta\,\mu_{1_2}\,G_2(Q_2 - \eta_2\,P_2)$$

$$\gamma\,\nu_2\,x_2\,G_2(1-\eta_2) + \gamma\,\nu_2\,x_1\,x_2\,\beta\big(Q_1 - P_1 + P_1\,E_2(1-\alpha_2)\big) = k\,x_1\,G_2(1-\eta_2) + k\,x_1\,x_2\,\beta\,\varepsilon_{1_2}\,E_2(Q_2 - \alpha_2\,P_2)$$

$$\nu_3\,x_4\,E_2(K_{32} - \alpha_2\,I_{32}) + \nu_3\,x_3\,x_4\,\beta\,\mu_{1_2}\,G_2(Q_{32} - \eta_2 P_{32}) = \nu_2\,x_3\,E_2(K_{32} - \alpha_2 I_{32}) + \nu_2\,x_3\,x_4\,\beta\,\mu_{1_3}\,G_2(K_{32} - \eta_2\,I_{32})\,\frac{Q_4 - \eta_3\,P_4}{1-\eta_3}$$

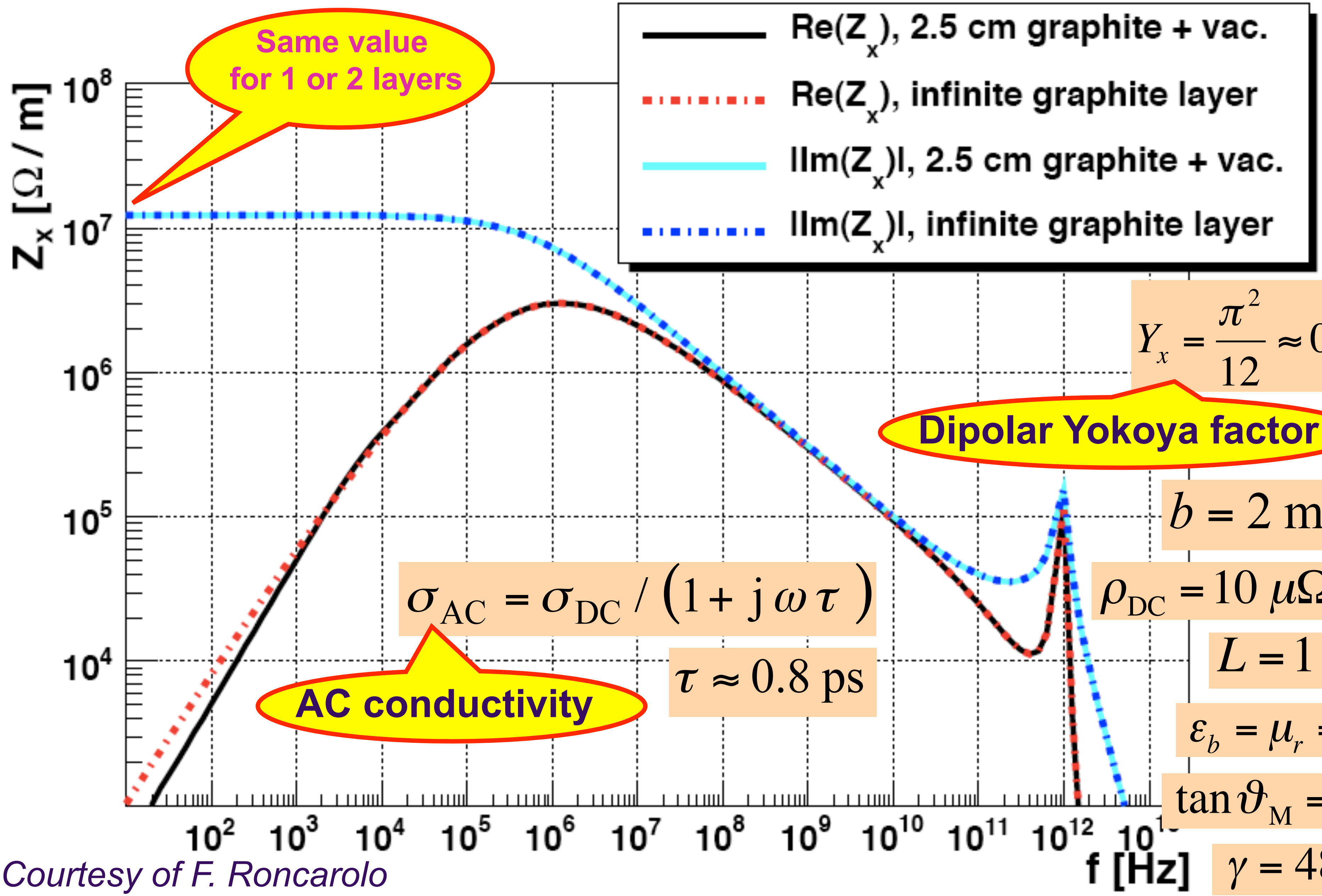
$$\nu_3\,x_4\,G_2(K_{32} - \eta_2\,I_{32}) + \nu_3\,x_3\,x_4\,\beta\,\varepsilon_{1_2}\,E_2(Q_{32} - \alpha_2 P_{32}) = \nu_2\,x_3\,G_2(K_{32} - \eta_2 I_{32}) + \nu_2\,x_3\,x_4\,\beta\,\varepsilon_{1_3}\,E_2(K_{32} - \alpha_2\,I_{32})\,\frac{Q_4 - \alpha_3\,P_4}{1-\alpha_3}$$

$$P_{1,2} = \frac{I_1'(x_{1,2})}{I_1(x_{1,2})} \qquad K_{32} = \frac{K_1(x_3)}{K_1(x_2)} \qquad x_{1,2} = \nu_{1,2}\,b \qquad Q_{32} = \frac{K_1'(x_3)}{K_1(x_2)} \qquad P_4 = \frac{I_1'(x_4)}{I_1(x_4)} \qquad \nu_{1,2,3} = k\,\sqrt{1 - \beta^2\,\varepsilon_{1_{1,2,3}}\,\mu_{1_{1,2,3}}}$$

$$Q_{1,2} = \frac{K_1'(x_{1,2})}{K_1(x_{1,2})} \qquad I_{32} = \frac{I_1(x_3)}{I_1(x_2)} \qquad x_3 = \nu_2\,d$$

$$x_4 = \nu_3\,d \qquad P_{32} = \frac{I_1'(x_3)}{I_1(x_2)} \qquad Q_4 = \frac{K_1'(x_4)}{K_1(x_4)} \qquad \alpha_3 = \eta_3 = 0$$

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (16/31)



Same value for 1 or 2 layers

$$Y_x = \frac{\pi^2}{12} \approx 0.8$$

Dipolar Yokoya factor

$$\sigma_{\mathrm{AC}} = \sigma_{\mathrm{DC}} \,/\, \left(1 + \mathrm{j}\,\omega\,\tau\right)$$

$$\tau \approx 0.8 \text{ ps}$$

AC conductivity

$$b = 2 \text{ mm}$$

$$\rho_{\mathrm{DC}} = 10\ \mu\Omega\mathrm{m}$$

$$L = 1 \text{ m}$$

$$\varepsilon_b = \mu_r = 1$$

$$\tan\vartheta_{\mathrm{M}} = 0$$

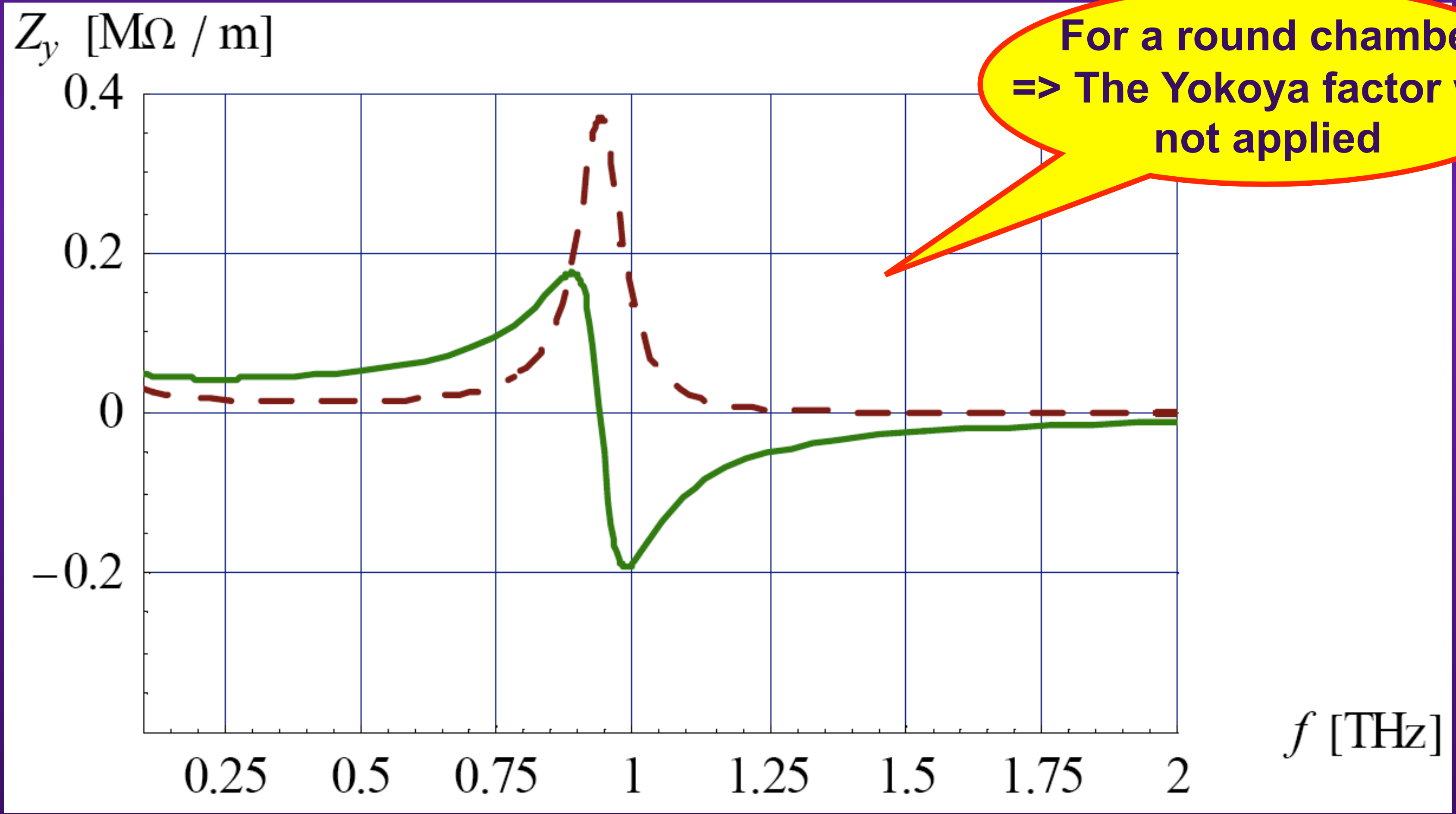
$$\gamma = 480$$

Courtesy of F. Roncarolo

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (17/31)

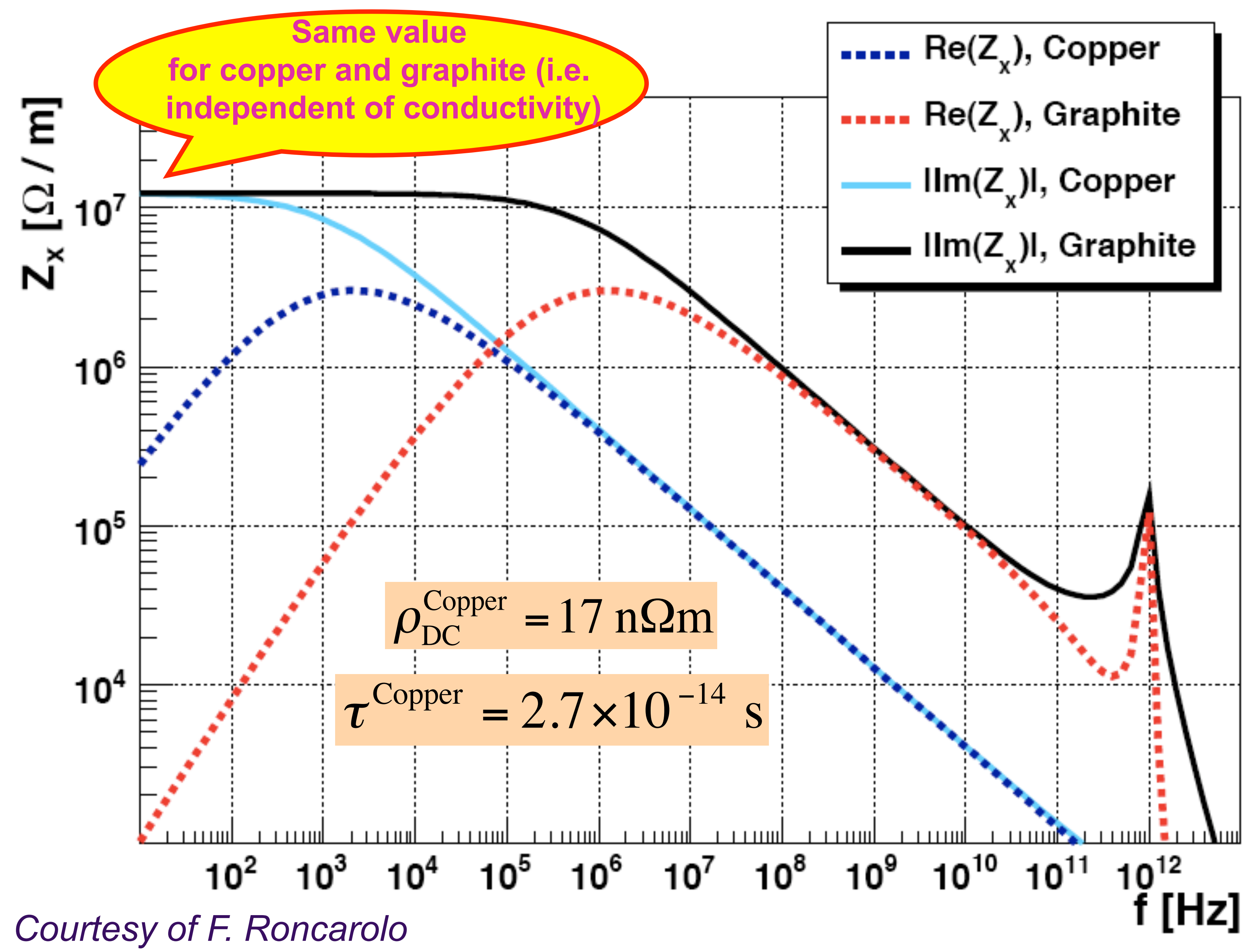
- On a linear plot, a resonance is clearly seen near 1 THz. The frequency of the resonance f_R is given by (when $(2\,\pi\, f_R\, \tau)^2 >> 1$, which is the case here)

$$f_R = \frac{1}{\pi\sqrt{2}}\left(\frac{Z_0\, c^3\, \sigma_{\mathrm{DC}}}{\tau\, b^2}\right)^{1/4} \approx 0.94 \text{ THz}$$



For a round chamber => The Yokoya factor was not applied

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (18/31)

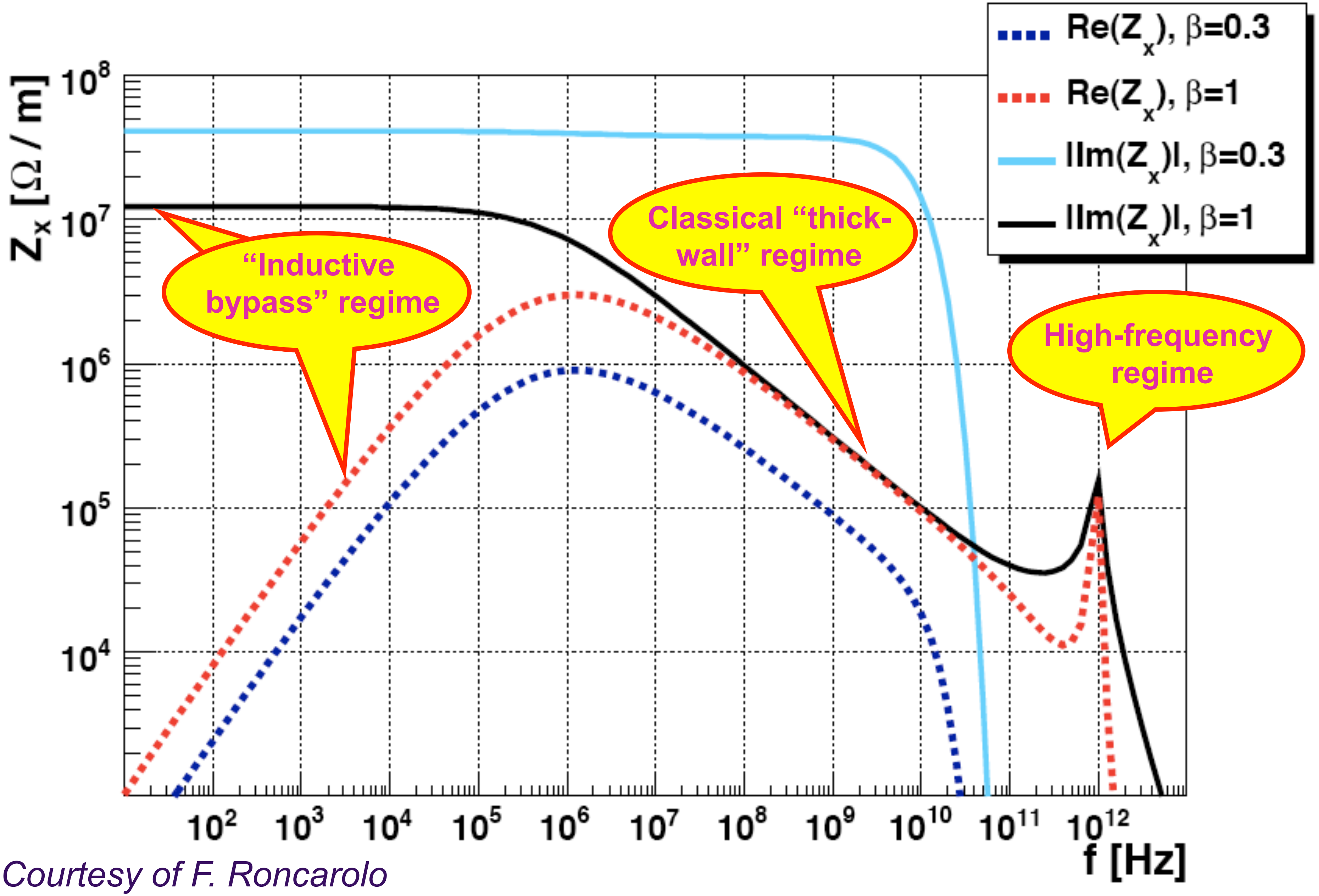


$$\rho_{DC}^{Copper} = 17 \text{ n}\Omega\text{m}$$

$$\tau^{Copper} = 2.7 \times 10^{-14} \text{ s}$$

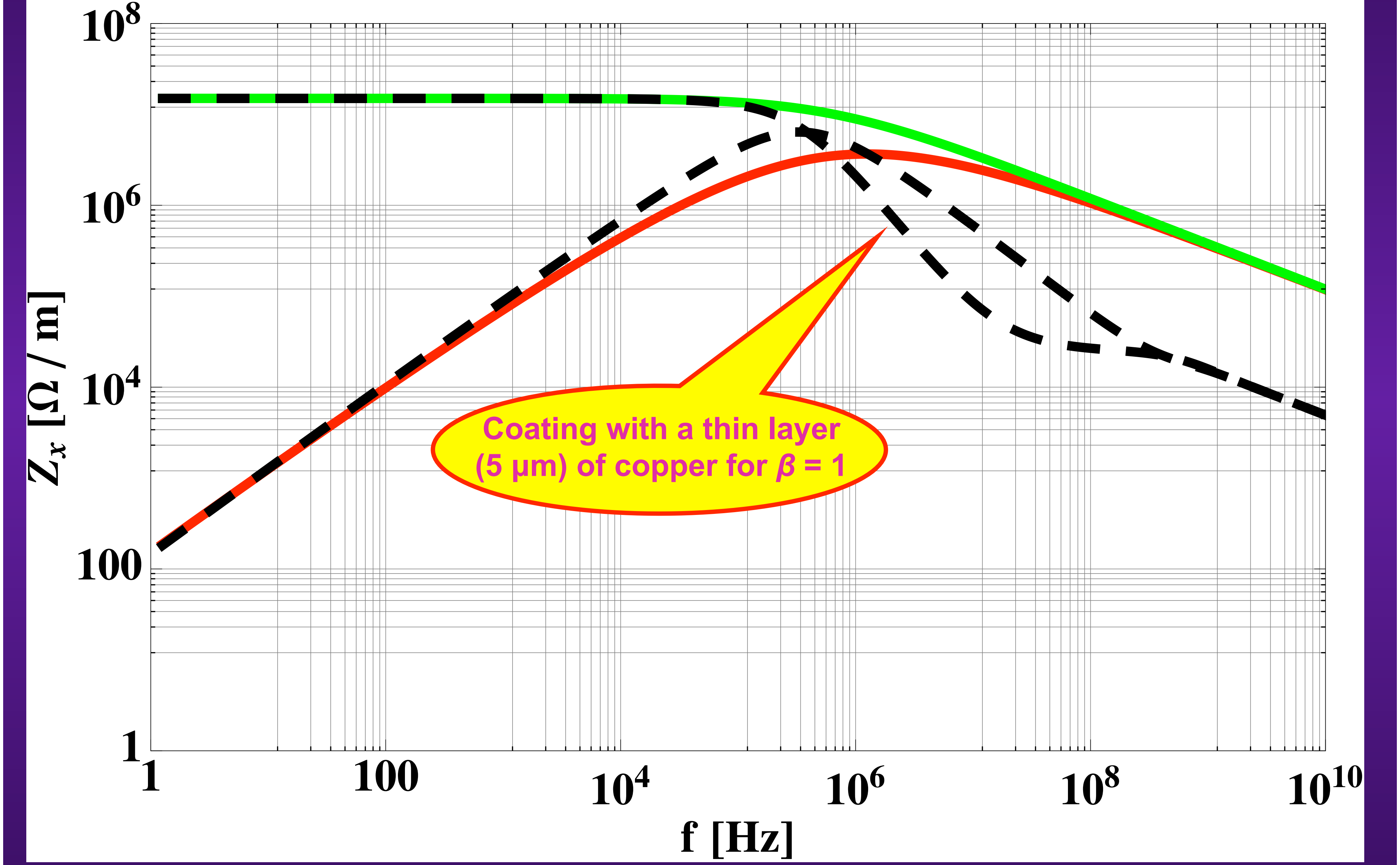
Courtesy of F. Roncarolo

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (19/31)



Courtesy of F. Roncarolo

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (20/31)



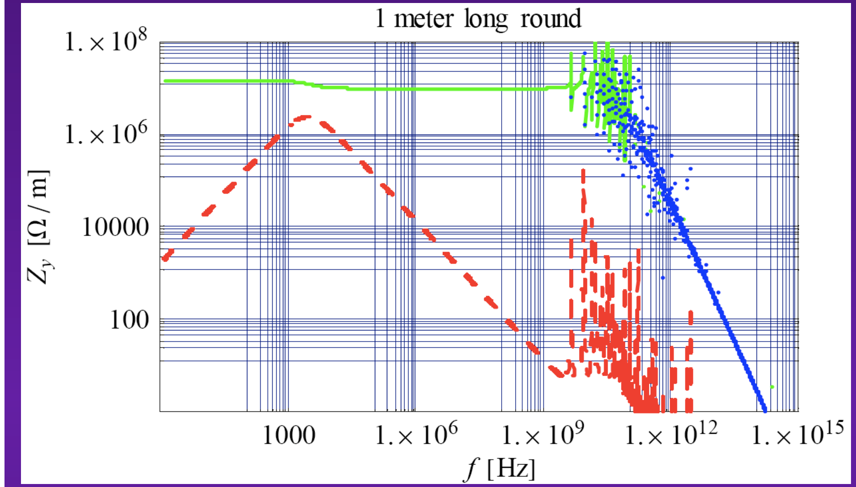
Coating with a thin layer (5 μm) of copper for β = 1

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (21/31)

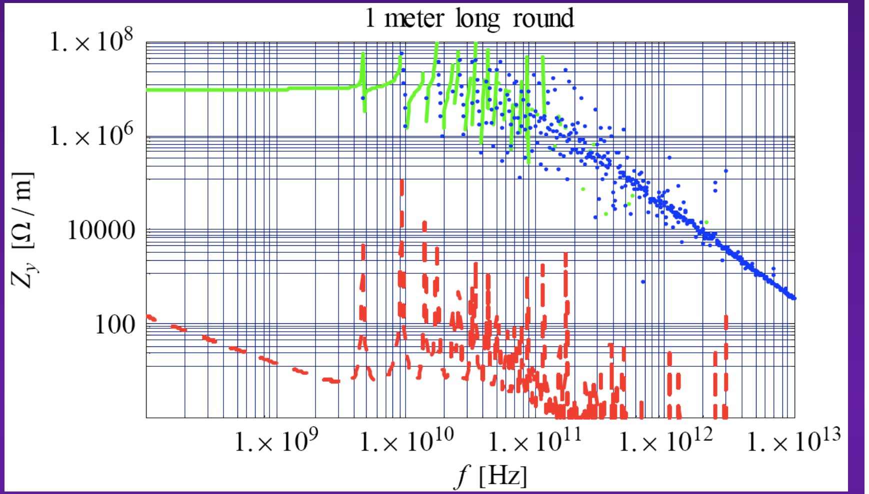
- **Example for a dielectric**

$\rho = 10^6\ \Omega\text{m}$ $\varepsilon_r' = 5$

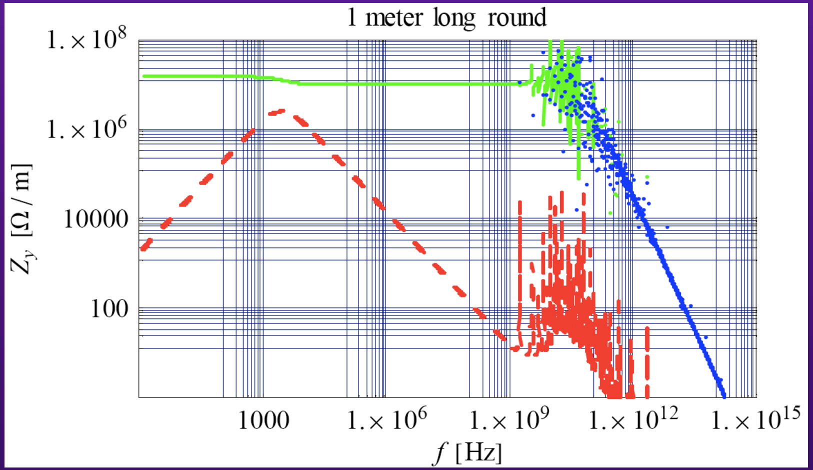
1 layer of thickness 1 cm and then a PC



=> Zoom



1 layer of thickness 2.5 cm and then a PC



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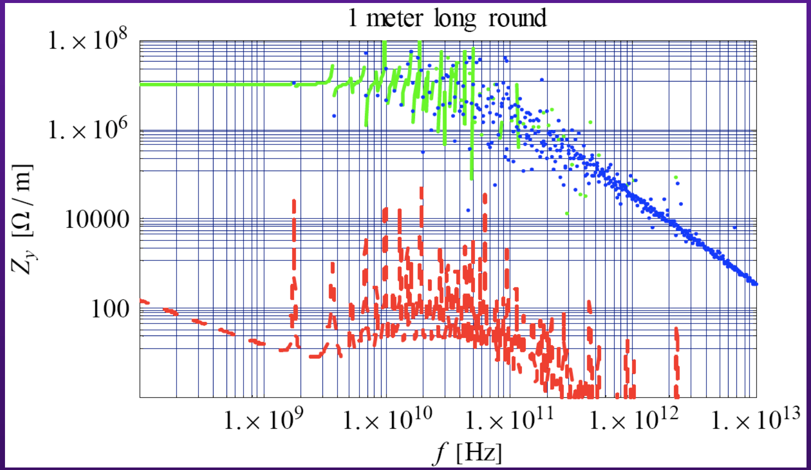
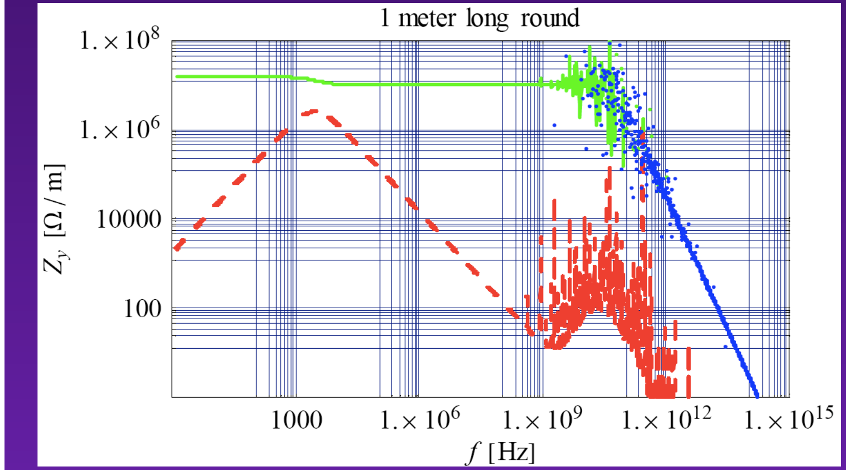
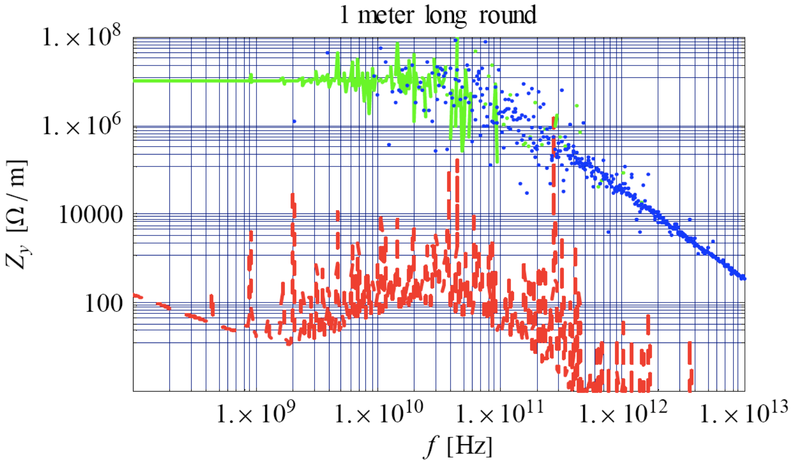


IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (22/31)

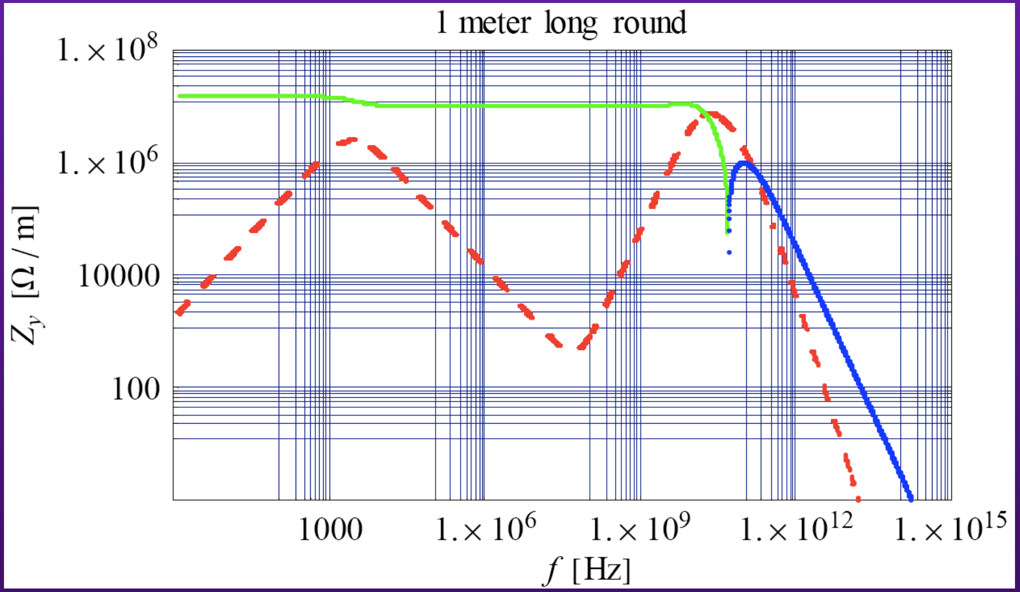
1 layer of thickness 10 cm and then a PC



=> Zoom



1 layer of infinite thickness



IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (23/31)

- **8) Approximate formula for the case of a LHC graphite collimator**

The interesting frequency range in the LHC lies between few kHz and few GHz. In this case a simple formula can be derived for a cylindrical geometry, which should be valid for any “relatively” good conductor with real permeability and the permittivity of vacuum. It can be written as (up to a certain frequency which depends on *β*)

$$\boxed{Z_x^{\mathrm{Wall}}(f) = \frac{j\,L\,Z_0}{2\pi\,b^2\,\beta\,\gamma^2} + \beta\,\frac{j\,L\,Z_0}{\pi\,b^2} \times \frac{1}{1 - \frac{x_2}{\mu_r} \times \frac{K_1'(x_2)}{K_1(x_2)}}}$$

with $\boxed{x_2 = (1+j)\,\frac{b}{\delta}}$ $\boxed{\delta = \sqrt{\frac{2}{\mu_0\,\mu_r\,\sigma\,\omega}}}$

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (24/31)

- Furthermore, this equation can be simplified even further in the two limiting cases using the following equations

$$\frac{K_1'(x_2)}{K_1(x_2)} = \begin{vmatrix} -\dfrac{1}{x_2} & \text{if } |x_2| << 1 \\ -1 & \text{if } |x_2| >> 1 \end{vmatrix}$$

- When $|x_2| << 1$, i.e. at very low frequency, the transverse "wall impedance" approaches a constant inductive value

$$\boxed{Z_x^{\text{Wall}}(f \to 0) = j\,\frac{L\,Z_0}{2\pi\,\beta\,b^2}}$$

for $\mu_r = 1$

Only electric images contribute as there are no ac magnetic images when f ➔ 0

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (25/31)

- When $\left| x_2 \right| >> 1$, the "classical thick-wall formula" is recovered (up to a certain frequency which depends on *β*)

$$Z_x^{\text{Wall}}\left(f \right) = \frac{j\,L\,Z_0}{2\pi\,b^2\,\beta\,\gamma^2} + \left(1 + j\right)\beta\,\frac{L\,Z_0\,\mu_r\,\delta}{2\,\pi\,b^3}$$

Coherent part (from the pipe) of the SC impedance => Electric images + ac magnetic images

Classical thick-wall formula for the "RW" impedance

- Note that the (broad) maximum of the real part of the transverse impedance is reached when $\text{Re}\left[\,x_2\,\right] \approx 1$, i.e. $\delta \approx b$, which means

$$f_{\max,\text{Re}} \approx \frac{\rho}{b^2} \times \frac{1}{\pi\,\mu_0}$$

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (26/31)

- 9) The same approach can be applied for the longitudinal plane (m = 0)

- 10) Longitudinal and transverse SC and RW impedances and wake fields in the “2nd” (“classical thick-wall”) frequency regime

  - PC = Perfectly Conductor wall

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (27/31)

$m = 0$

Used to compute the longitudinal impedance

$$E_\vartheta^{PC0} = B_r^{PC0} = B_s^{PC0} = 0$$

$$E_s^{PC0} = -\frac{Q}{2\pi\varepsilon_0\gamma^2}\ln\left(\frac{b}{r}\right)\delta'(s-\upsilon t)$$

$$E_r^{PC0} = \frac{Q}{2\pi\varepsilon_0 r}\delta(s-\upsilon t)$$

$$B_\vartheta^{PC0} = \frac{\beta}{c}E_r^{PC0}$$

$m = 1$

Used to compute the transverse impedance

$$B_s^{PC1} = 0$$

$$E_s^{PC1} = \frac{Q_1\cos(\vartheta)}{2\pi\varepsilon_0\gamma^2}\left[\frac{1}{r} - \frac{r}{b^2}\right]\delta'(s-\upsilon t)$$

$$E_r^{PC1} = \frac{Q_1\cos(\vartheta)}{2\pi\varepsilon_0}\left[\frac{1}{r^2} + \frac{1}{b^2}\right]\delta(s-\upsilon t)$$

$$E_\vartheta^{PC1} = \frac{Q_1\sin(\vartheta)}{2\pi\varepsilon_0}\left[\frac{1}{r^2} - \frac{1}{b^2}\right]\delta(s-\upsilon t)$$

$$B_\vartheta^{PC1} = \frac{\beta}{c}E_r^{PC1} \qquad B_r^{PC1} = -\frac{\beta}{c}E_\vartheta^{PC1}$$

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (28/31)

Force on a particle with charge *q*

$$\vec{F} = q \left[E_s \, \vec{s} + \left(E_r - \upsilon \, B_\vartheta \right) \vec{r} + \left(E_\vartheta + \upsilon \, B_r \right) \vec{\vartheta} \right]$$

***m* = 0**

$$F_s^{PC0} = - \frac{q \, Q}{2 \pi \varepsilon_0 \gamma^2} \ln \left(\frac{b}{r} \right) \delta' \left(s - \upsilon \, t \right)$$

$$F_r^{PC0} = \frac{q \, Q}{2 \pi \varepsilon_0 \, r \, \gamma^2} \delta \left(s - \upsilon \, t \right)$$

$$F_\vartheta^{PC0} = 0$$

***m* = 1**

$$F_s^{PC1} = \frac{q \, Q_1 \cos(\vartheta)}{2 \pi \varepsilon_0 \gamma^2} \left[\frac{1}{r} - \frac{r}{b^2} \right] \delta' \left(s - \upsilon \, t \right)$$

$$F_r^{PC1} = \frac{q \, Q_1 \cos(\vartheta)}{2 \pi \varepsilon_0 \gamma^2} \left[\frac{1}{r^2} + \frac{1}{b^2} \right] \delta \left(s - \upsilon \, t \right)$$

$$F_\vartheta^{PC1} = \frac{q \, Q_1 \sin(\vartheta)}{2 \pi \varepsilon_0 \gamma^2} \left[\frac{1}{r^2} - \frac{1}{b^2} \right] \delta \left(s - \upsilon \, t \right)$$

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (29/31)

***m* = 0**

$$Z_{//}^{PC0}(\omega) = -j\,\frac{L\,\omega\,Z_0}{2\pi\,c\,\beta^2\gamma^2}\ln\left(\frac{b}{a}\right)$$

$$W_{//}^{PC0}(t) = \frac{L\,Z_0}{2\pi\,c\,\beta^2\gamma^2}\ln\left(\frac{b}{a}\right)\delta'(t)$$

Behind the bunch

For L = 2 π R

$$Z_{//}^{PC0}(\omega) = -j\,\frac{\omega\,Z_0}{\omega_0\,\beta\,\gamma^2}\ln\left(\frac{b}{a}\right)$$

$$W_{//}^{PC0}(t) = \frac{Z_0}{\omega_0\,\beta\,\gamma^2}\ln\left(\frac{b}{a}\right)\delta'(t)$$

***m* = 1**

$$Z_{\perp}^{PC1}(\omega) = -j\,\frac{L\,Z_0}{2\pi\,\beta\,\gamma^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)$$

$$W_{\perp}^{PC1}(t) = \frac{L\,Z_0}{2\pi\,\beta\,\gamma^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\delta(t)$$

For L = 2 π R

$$Z_{\perp}^{PC1}(\omega) = -j\,\frac{R\,Z_0}{\beta\,\gamma^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)$$

$$W_{\perp}^{PC1}(t) = \frac{R\,Z_0}{\beta\,\gamma^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\delta(t)$$

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (30/31)

- **Resistive object (with β = 1)**

***m* = 0**

$$F_s^{RW0} = \frac{q\,Q\,c\,\sqrt{Z_0}}{4\pi^{3/2}\,b\,\sqrt{\sigma}\,|z|^{3/2}}$$

$$F_r^{RW0} = F_\vartheta^{RW0} = 0$$

***m* = 1**

$$F_s^{RW1} = \frac{q\,Q_1\cos(\vartheta)\,c\,r\,\sqrt{Z_0}}{2\pi^{3/2}\,b^3\,\sqrt{\sigma}\,|z|^{3/2}}$$

$$F_r^{RW1} = \frac{q\,Q_1\cos(\vartheta)\,c\,\sqrt{Z_0}}{\pi^{3/2}\,b^3\,\sqrt{\sigma}\,|z|^{1/2}}$$

$$F_\vartheta^{RW1} = -\frac{q\,Q_1\sin(\vartheta)\,c\,\sqrt{Z_0}}{\pi^{3/2}\,b^3\,\sqrt{\sigma}\,|z|^{1/2}}$$

IMPEDANCE OF AN INFINITELY LONG SMOOTH BEAM PIPE (31/31)

***m* = 0**

$$Z_{//}^{RW0}(\omega) = (1 + j) \frac{L}{2 \pi b} \sqrt{\frac{\omega Z_0}{2 c \sigma}}$$

$$W_{//}^{RW0}(t) = - \frac{L}{4 \pi^{3/2} b} \sqrt{\frac{Z_0}{c \sigma}} \times \frac{1}{t^{3/2}}$$

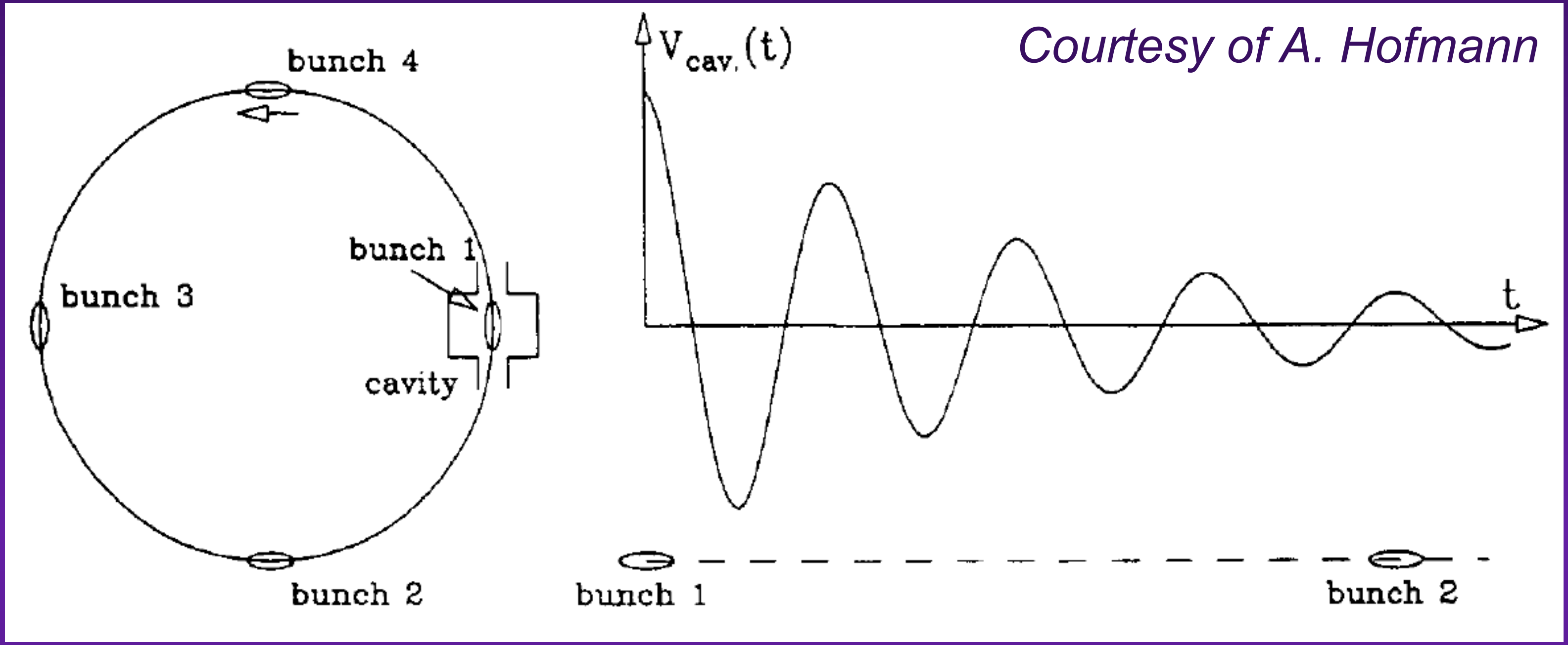
***m* = 1**

$$Z_{\perp}^{RW1}(\omega) = (1 + j) \frac{L Z_0}{\pi b^3} \frac{1}{\sqrt{2 \mu_0 \sigma \omega}}$$

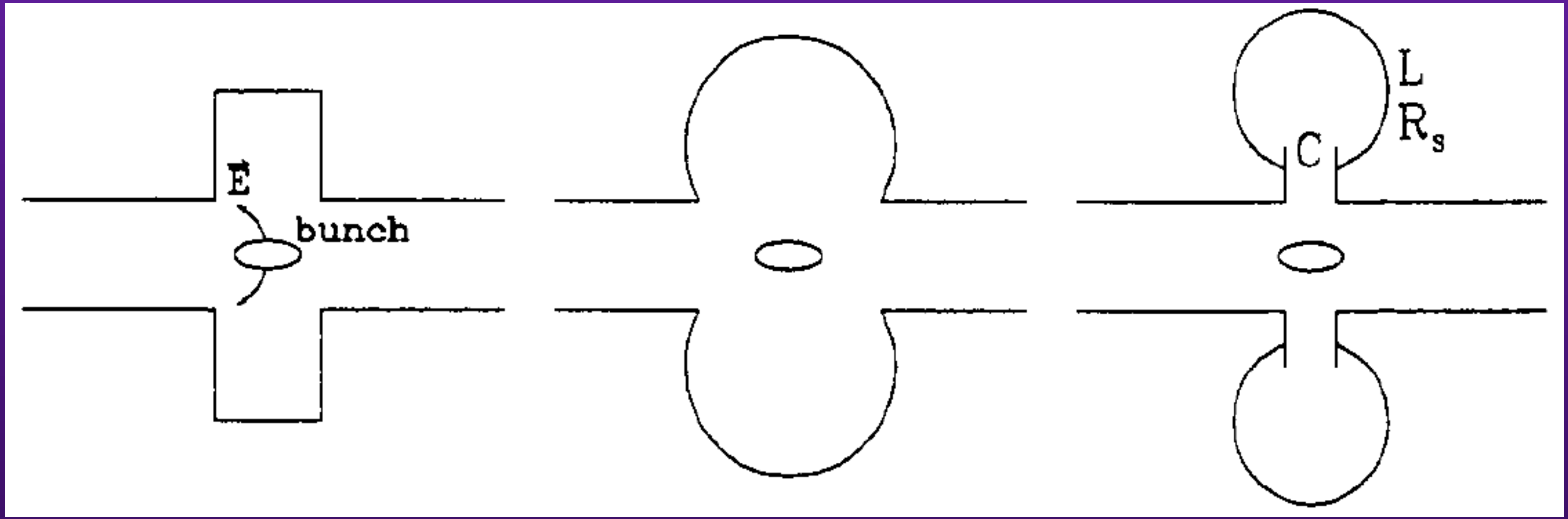
$$W_{\perp}^{RW1}(t) = - \frac{L}{\pi^{3/2} b^3} \sqrt{\frac{c Z_0}{\sigma}} \times \frac{1}{t^{1/2}}$$

IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (1/25)

CAVITY RESONANCE

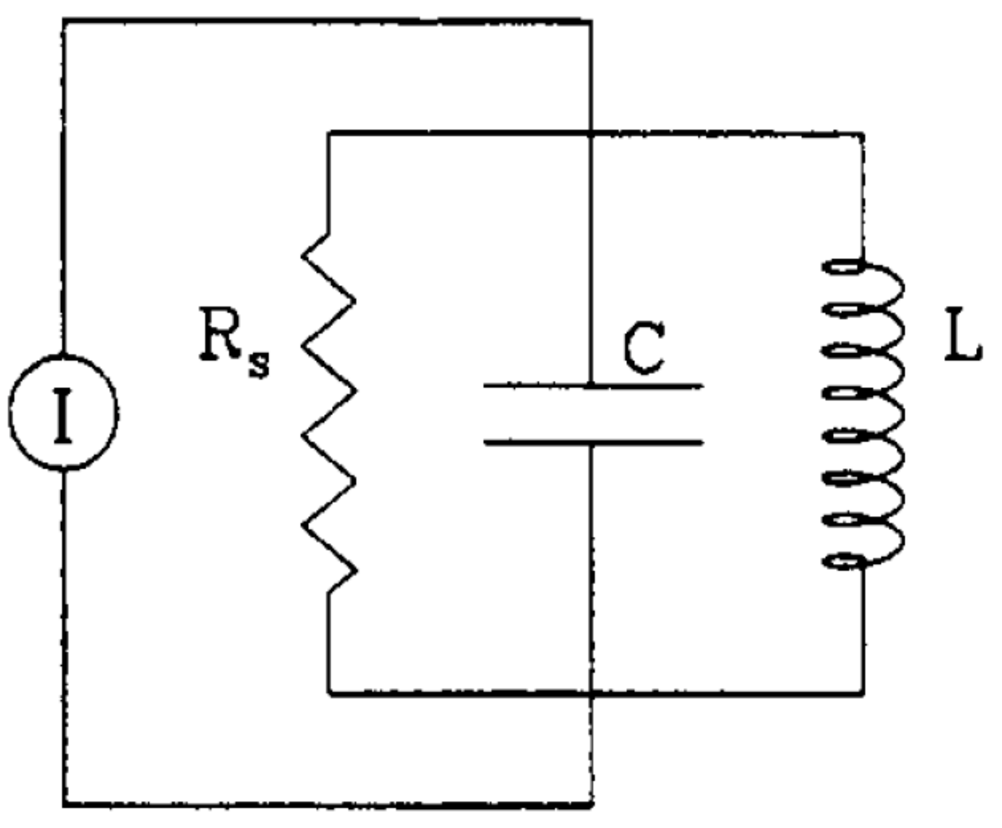


Courtesy of A. Hofmann



IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (2/25)

- **RLC circuit equivalent to a cavity resonance**



R_s = Shunt impedance

C = Capacity

L = Inductance

IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (3/25)

- In a real cavity, these 3 parameters cannot easily be separated => We use some other related parameters which can be measured directly

$$\boxed{\omega_r = \frac{1}{\sqrt{L\,C}}}$$

= Resonance (angular) frequency

$$\boxed{Q = R_s\,\sqrt{\frac{C}{L}} = \frac{R_s}{L\,\omega_r} = R_s\,C\,\omega_r}$$

= Quality factor

$$\boxed{\alpha = \frac{\omega_r}{2\,Q}}$$

= Damping rate

IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (4/25)

- **If this circuit is driven by a current *I*, the voltages across each element are**

$$V_r = R_s \, I_R$$

$$V_C = \frac{1}{C} \int I_C \, dt$$

$$V_L = L \, \frac{d \, I_L}{d \, t}$$

$$V = V_R = V_C = V_L$$

$$I = I_R + I_C + I_L$$

IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (5/25)

=> $$\dot{I} = \dot{I}_R + \dot{I}_C + \dot{I}_L = \frac{\dot{V}_R}{R_s} + C\,\ddot{V}_C + \frac{V_L}{L}$$

=> $$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2\,V = \frac{\omega_r\,R_s}{Q}\,\dot{I}$$

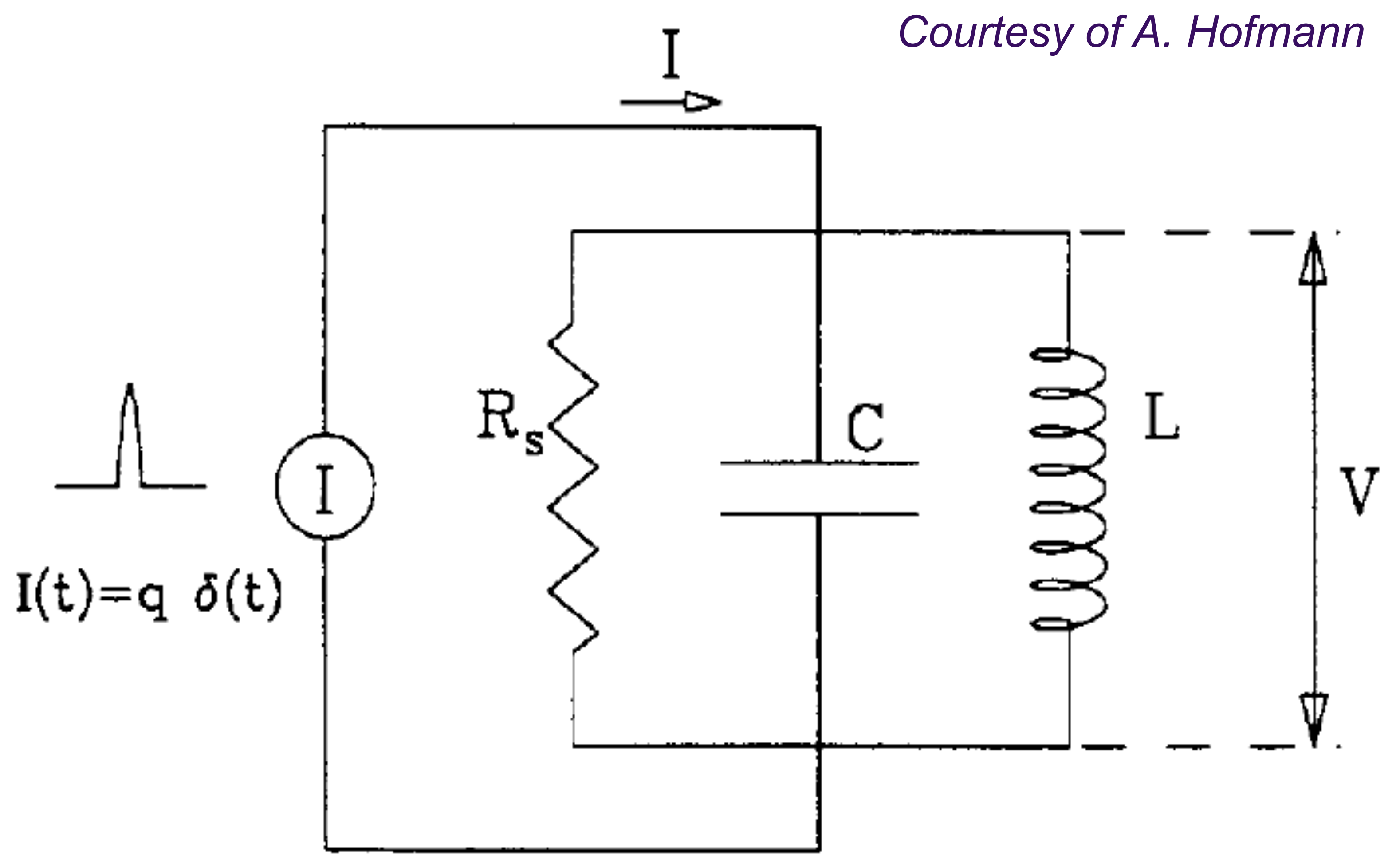
The solution of the homogeneous equation is a damped oscillation

$$V(t) = \hat{V}\,e^{-\alpha t}\cos\left[\omega_r\sqrt{1-\frac{1}{4\,Q^2}}\;t+\phi\right]$$

or $$V(t) = e^{-\alpha t}\left\{A\cos\left[\omega_r\sqrt{1-\frac{1}{4\,Q^2}}\;t\right] + B\sin\left[\omega_r\sqrt{1-\frac{1}{4\,Q^2}}\;t\right]\right\}$$

IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (6/25)

- **Response of the RLC circuit (representing a cavity) to a δ-function pulse (= very short bunch) at time t = 0**



Courtesy of A. Hofmann

IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (7/25)

The charge *q* induces a voltage in the capacity

$$V\left(0^{+}\right)=\frac{q}{C}=\frac{\omega_{r}\,R_{s}}{Q}\,q$$

The energy stored in the capa (= energy lost by the charge) is

$$U=\frac{1}{2}\,C\,V^{2}\left(0^{+}\right)=\frac{q^{2}}{2\,C}=\frac{\omega_{r}\,R_{s}}{2\,Q}\,q^{2}=\frac{V\left(0^{+}\right)}{2}\,q=k_{pm}\,q^{2}$$

Parasitic loss mode factor

IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (8/25)

The charged capa will now discharge 1st through the resistor and then also through the inductance

$$\dot{V}\left(0^+\right) = \frac{\dot{q}}{C} = \frac{I_R}{C} = -\frac{V\left(0^+\right)}{C\,R_s} = -\frac{\omega_r^2\,R_s}{Q^2}\,q = -\frac{2\,\omega_r\,k_{pm}}{Q}\,q$$

The voltage in this resonant circuit has now the initial conditions

$$V\left(0^+\right) = 2\,k_{pm}\,q = A$$

$$\dot{V}\left(0^+\right) = -\frac{2\,\omega_r\,k_{pm}}{Q}\,q = B\,\overline{\omega}_r - \alpha\,A$$

$$\overline{\omega}_r = \omega_r\,\sqrt{1-\frac{1}{4\,Q^2}}$$

IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (9/25)

=> $$\boxed{A = 2\, k_{pm}\, q}$$

$$\boxed{B = -\frac{A}{2\,Q\sqrt{1-\dfrac{1}{4\,Q^2}}}}$$

=> $$\boxed{V(t) = 2\, k_{pm}\, q\, e^{-\alpha t}\left[\cos\left(\overline{\omega}_r\, t\right) - \frac{\sin\left(\overline{\omega}_r\, t\right)}{2\,Q\sqrt{1-\dfrac{1}{4\,Q^2}}}\right]}$$

IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (10/25)

=> A 2nd point charge *q'* going through the cavity at a later time t will gain or lose the energy

$$\boxed{U = q' \, V(t)}$$

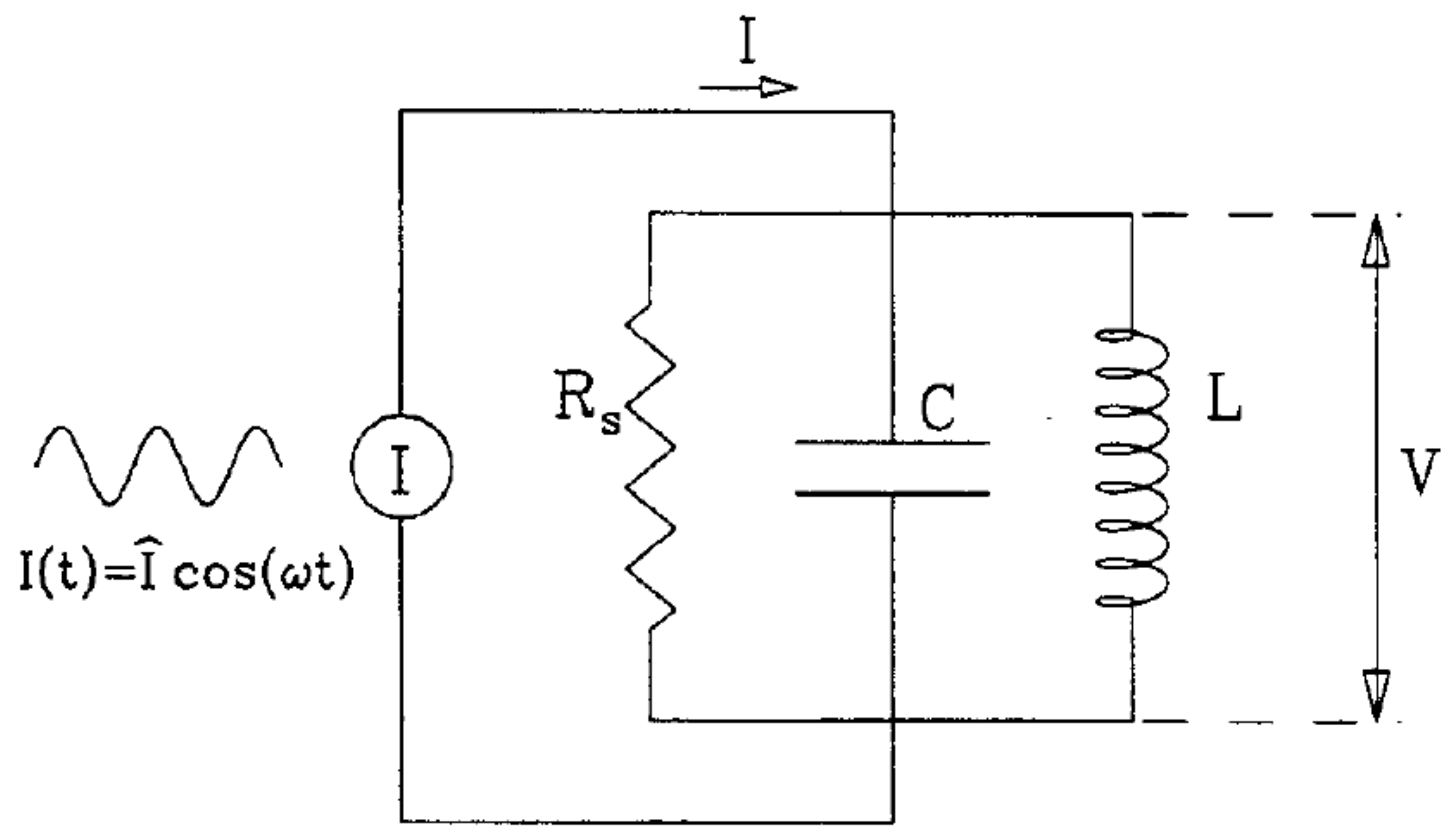
This energy gain/loss per unit source and unit test (probe) charge is called the wake potential of a point charge or also the Green function G(t)

$$\boxed{G(t) = \frac{U}{q\,q'} = \frac{V(t)}{q} = 2\,k_{pm}\,e^{-\alpha t}\left[\cos\left(\overline{\omega}_r\,t\right) - \frac{\sin\left(\overline{\omega}_r\,t\right)}{2\,Q\sqrt{1-\frac{1}{4\,Q^2}}}\right]}$$

When $\boxed{Q >> 1}$, it yields $\boxed{G(t) = 2\,k_{pm}\,e^{-\alpha t}\cos\left(\overline{\omega}_r\,t\right)}$

IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (11/25)

- **Response of the RLC circuit (representing a cavity) to a harmonic excitation**



IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (12/25)

$$\ddot{V} + \frac{\omega_r}{Q} \dot{V} + \omega_r^2 \, V = \frac{\omega_r \, R_s}{Q} \, \dot{I}$$

$$\dot{I} = - \hat{I} \, \omega \sin\left(\omega \, t \right)$$

The solution of the homogeneous equation is a damped oscillation which disappears after some time. We are left with the particular solution

$$V\left(t \right) = A \cos\left(\omega \, t \right) + B \sin\left(\omega \, t \right)$$

IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (13/25)

=> $$A = \frac{R_s \, \hat{I}}{1 + Q^2 \left(\frac{\omega_r^2 - \omega^2}{\omega_r \, \omega} \right)^2}$$

$$B = - \, A \, Q \left(\frac{\omega_r^2 - \omega^2}{\omega_r \, \omega} \right)$$

In phase with excitation
=> Can absorb energy
=> Resistive term

Out of phase with excitation
=> Cannot absorb energy
=> Reactive term

=> $$V(t) = R_s \, \hat{I} \, \frac{\cos(\omega \, t) - Q \left(\frac{\omega_r^2 - \omega^2}{\omega_r \, \omega} \right) \sin(\omega \, t)}{1 + Q^2 \left(\frac{\omega_r^2 - \omega^2}{\omega_r \, \omega} \right)^2}$$

IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (14/25)

- Complex notations (involving positive and negative frequencies, as opposed to the only positive frequencies used before)

$$I = \hat{I}\, e^{j\omega t}$$

Looking for a particular solution (of the differential equation) of the form $V(t) = V_0\, e^{j\omega t}$, yields the impedance

$$Z(\omega) = \frac{V_0}{\hat{I}} = \frac{R_s}{1 + j\,Q\left(\dfrac{\omega}{\omega_r} - \dfrac{\omega_r}{\omega}\right)} = Z_R(\omega) + j\,Z_I(\omega)$$

IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (15/25)

For a large quality factor, the impedance is only large for $\omega \approx \omega_r$

or $\dfrac{\left|\omega - \omega_r\right|}{\omega_r} = \dfrac{\left|\Delta\omega\right|}{\omega_r} << 1$

$$Z(\omega) \approx R_s \frac{1 - j\,2\,Q\,\dfrac{\Delta\omega}{\omega_r}}{1 + 4\,Q^2\left(\dfrac{\Delta\omega}{\omega_r}\right)^2}$$

IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (16/25)

- One can check that (using the useful relations in “Introduction”)

$$G_m^{//}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Z_m^{//}(\omega)\, e^{j\omega t}\, d\omega$$

$$Z_m^{//}(\omega) = \int_{-\infty}^{+\infty} G_m^{//}(t)\, e^{-j\omega t}\, dt = \int_{0}^{+\infty} G_m^{//}(t)\, e^{-j\omega t}\, dt$$

As there is no field before the particles arrive

$$Z_m^{//}(\omega) = \frac{R_s}{1 + j\, Q \left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega} \right)}$$

$$\overline{\omega}_r = \omega_r \sqrt{1 - \frac{1}{4\, Q^2}}$$

$$G_m^{//}(t) = \frac{\omega_r\, R_s}{Q}\, e^{-\alpha t} \left[\cos\left(\overline{\omega}_r\, t\right) - \frac{\alpha}{\overline{\omega}_r} \sin\left(\overline{\omega}_r\, t\right) \right]$$

$$\alpha = \frac{\omega_r}{2\, Q}$$

IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (17/25)

- **The Panofsky-Wenzel theorem requires that the same resonator also gives a transverse impedance**

$$\boxed{Z_m^{\perp}(\omega) = \frac{c}{\omega} Z_m^{//}(\omega) = \frac{c}{\omega} \frac{R_s}{1 + j\,Q\left(\dfrac{\omega}{\omega_r} - \dfrac{\omega_r}{\omega}\right)} = \frac{\omega_r}{\omega} \frac{R_{\perp}}{1 + j\,Q\left(\dfrac{\omega}{\omega_r} - \dfrac{\omega_r}{\omega}\right)}}$$

with $\boxed{R_{\perp} = \dfrac{c}{\omega_r} R_s}$

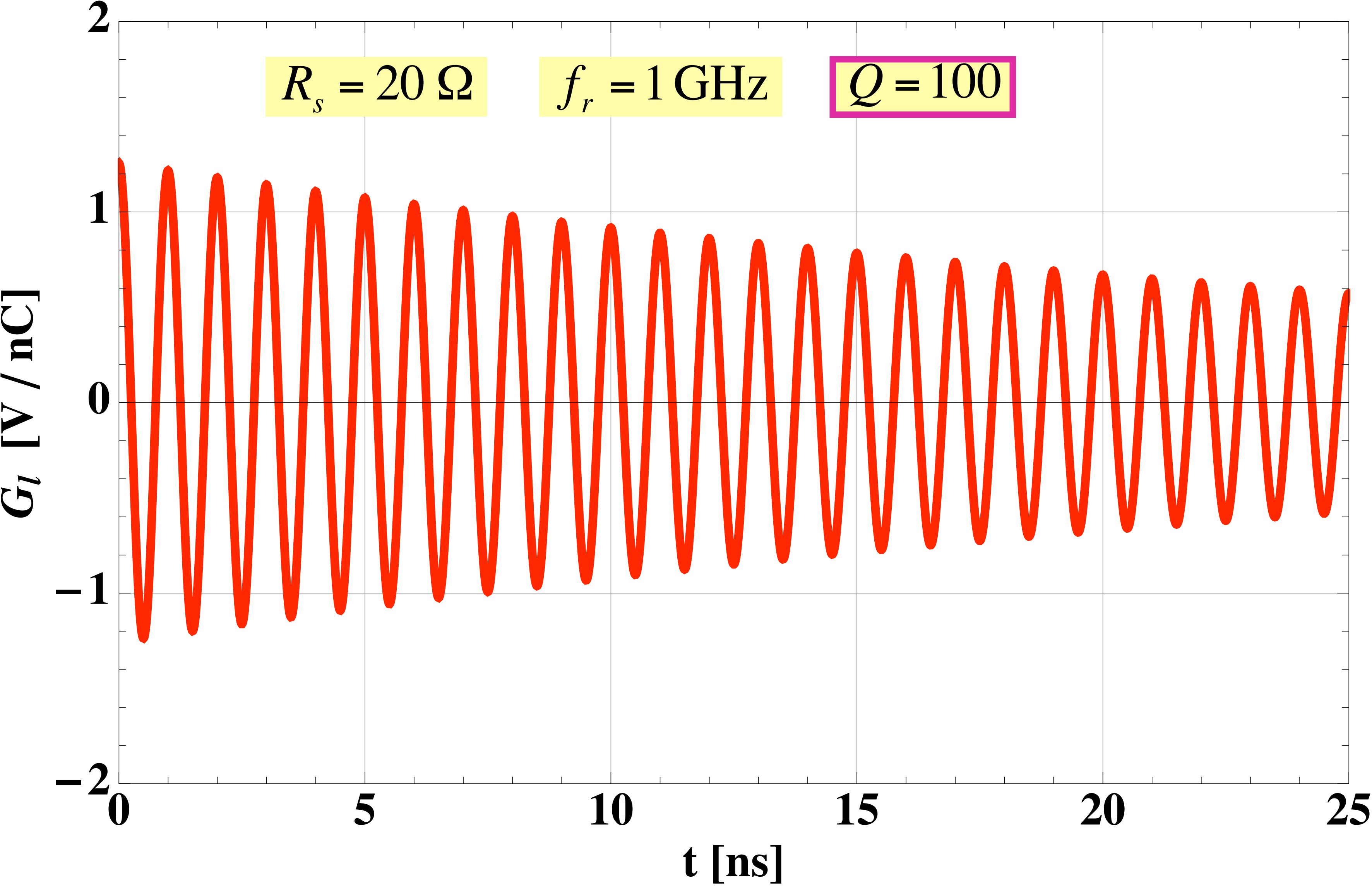
$$G_m^{\perp}(t) = \frac{j}{2\pi} \int_{-\infty}^{+\infty} Z_m^{\perp}(\omega)\, e^{j\omega t}\, d\omega$$

=>

$$\boxed{G_m^{\perp}(t) = \frac{\omega_r^2 R_{\perp}}{Q\,\overline{\omega}_r} e^{-\alpha t} \sin\left(\overline{\omega}_r\, t\right)}$$

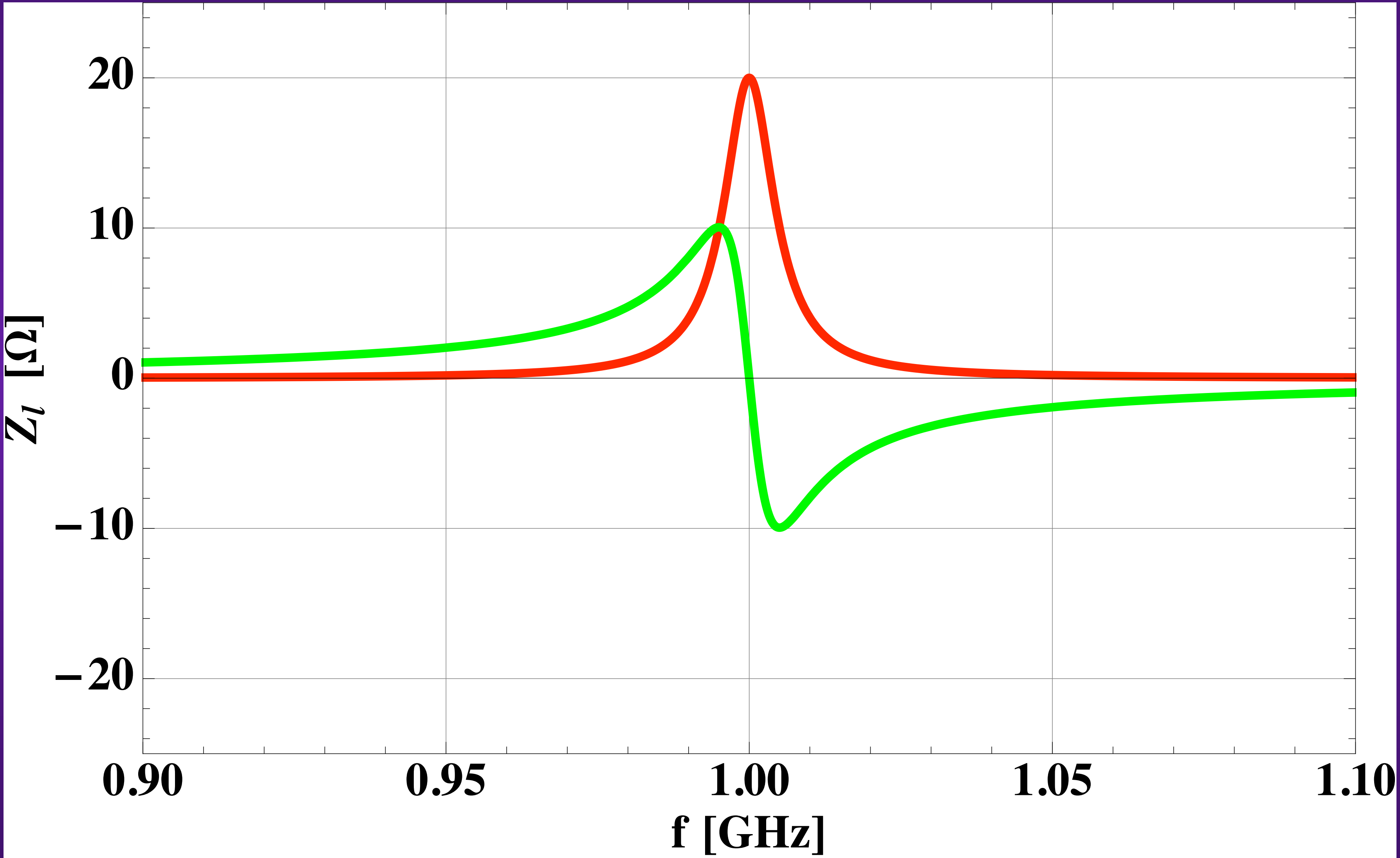
IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (18/25)

- **Example 1 in the longitudinal plane (resonator wake field)**



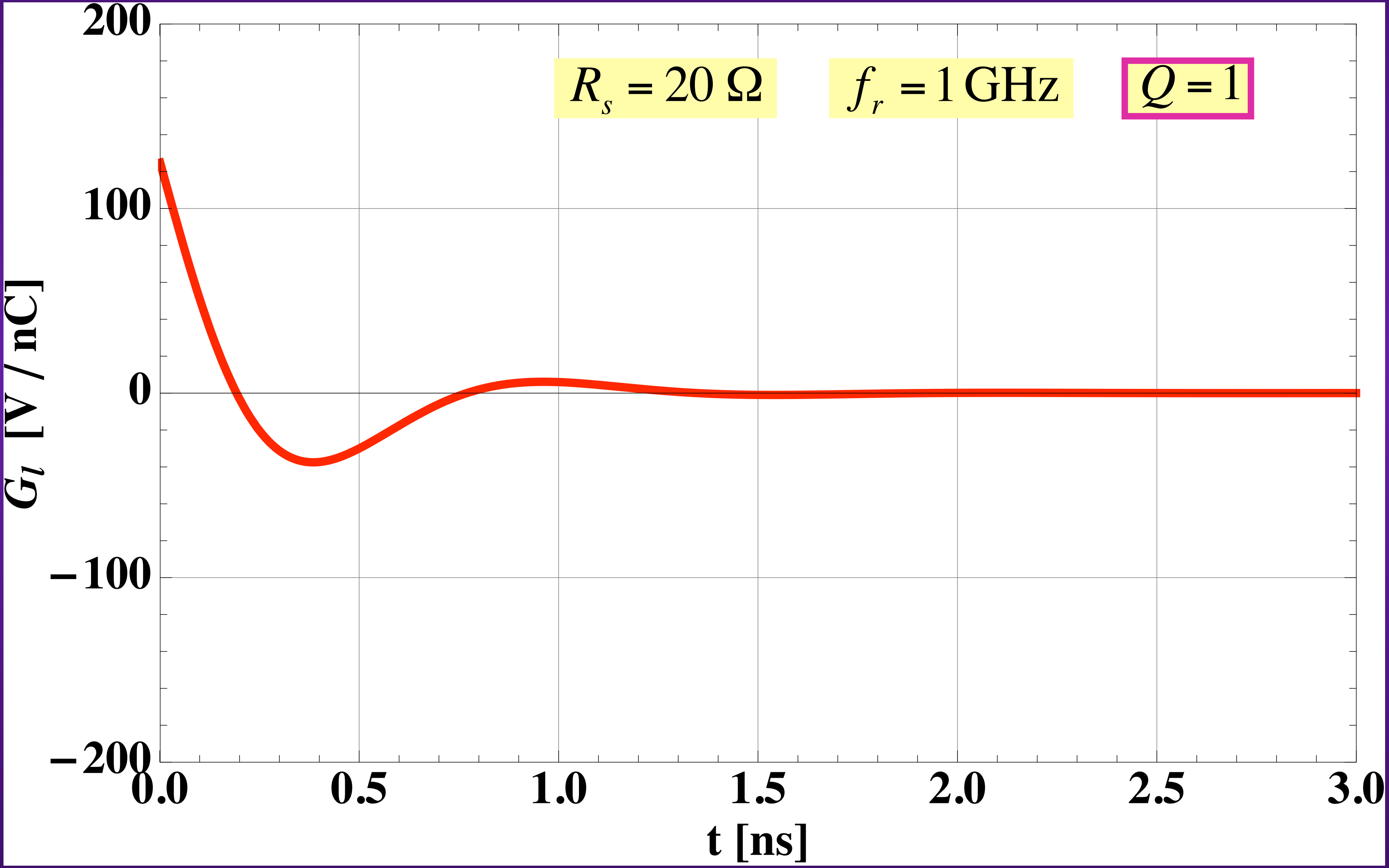
IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (19/25)

- **Example 1 in the longitudinal plane (resonator impedance)**



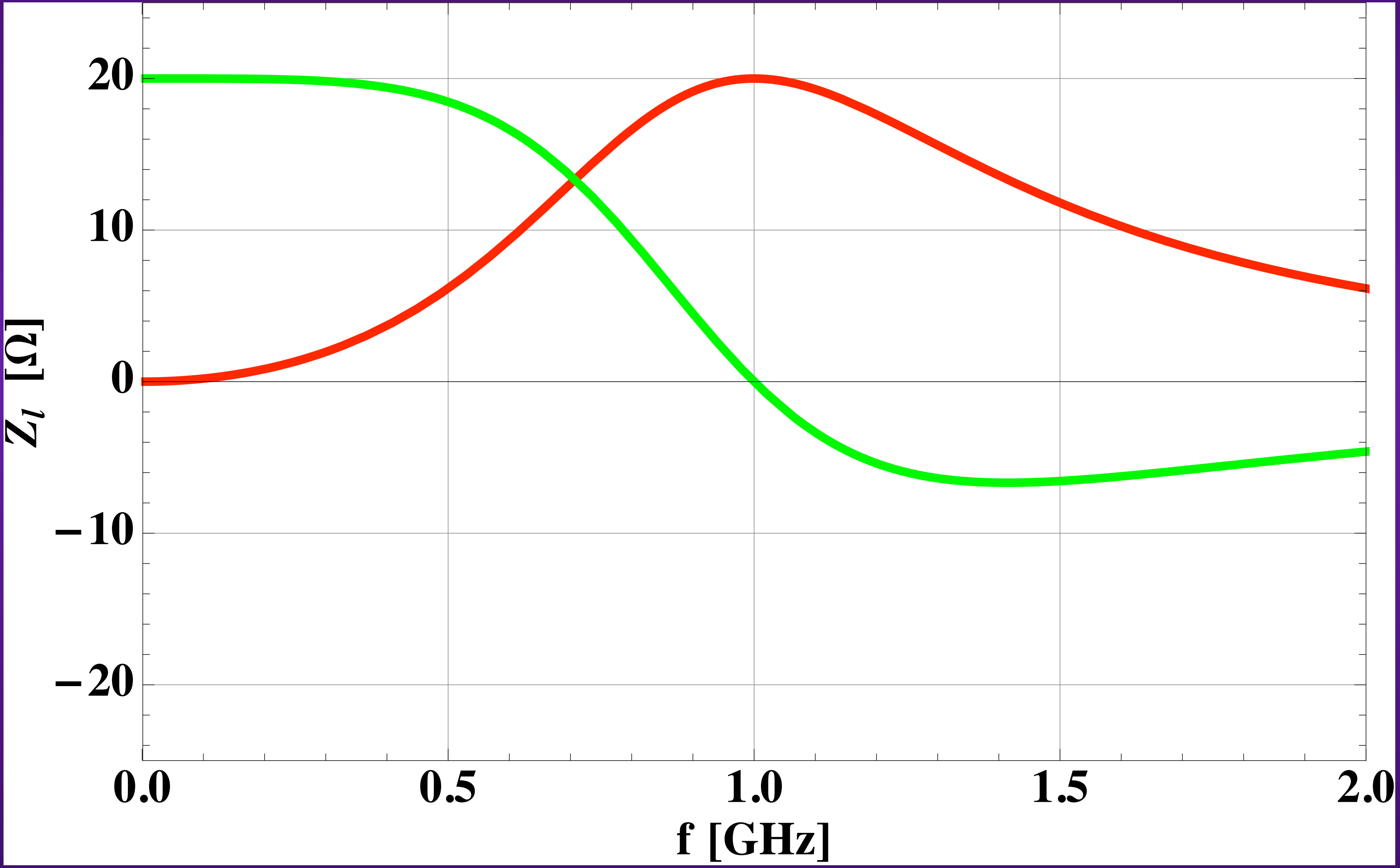
IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (20/25)

- **Example 2 in the longitudinal plane (resonator wake field)**



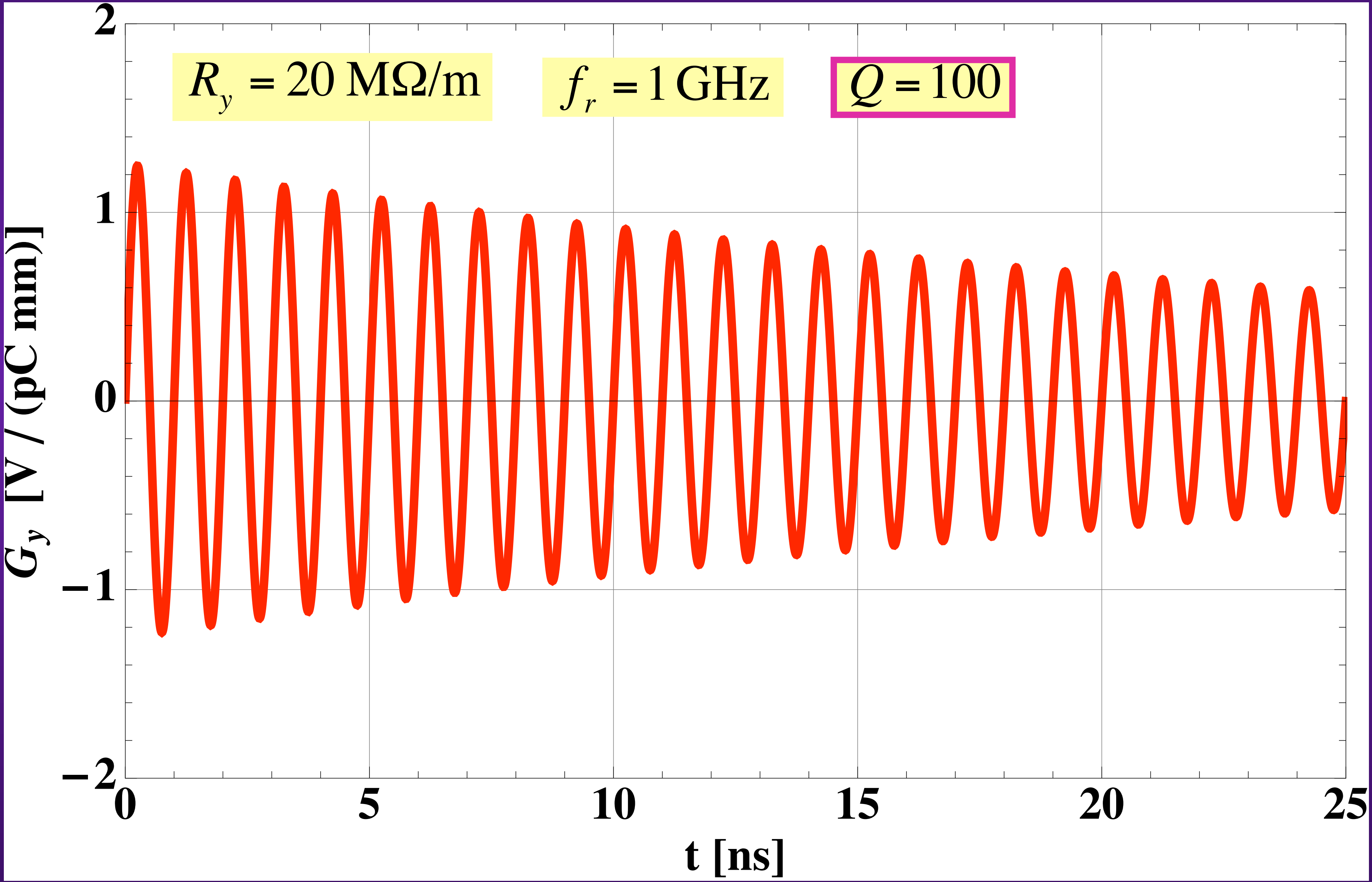
IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (21/25)

- **Example 2 in the longitudinal plane (resonator impedance)**



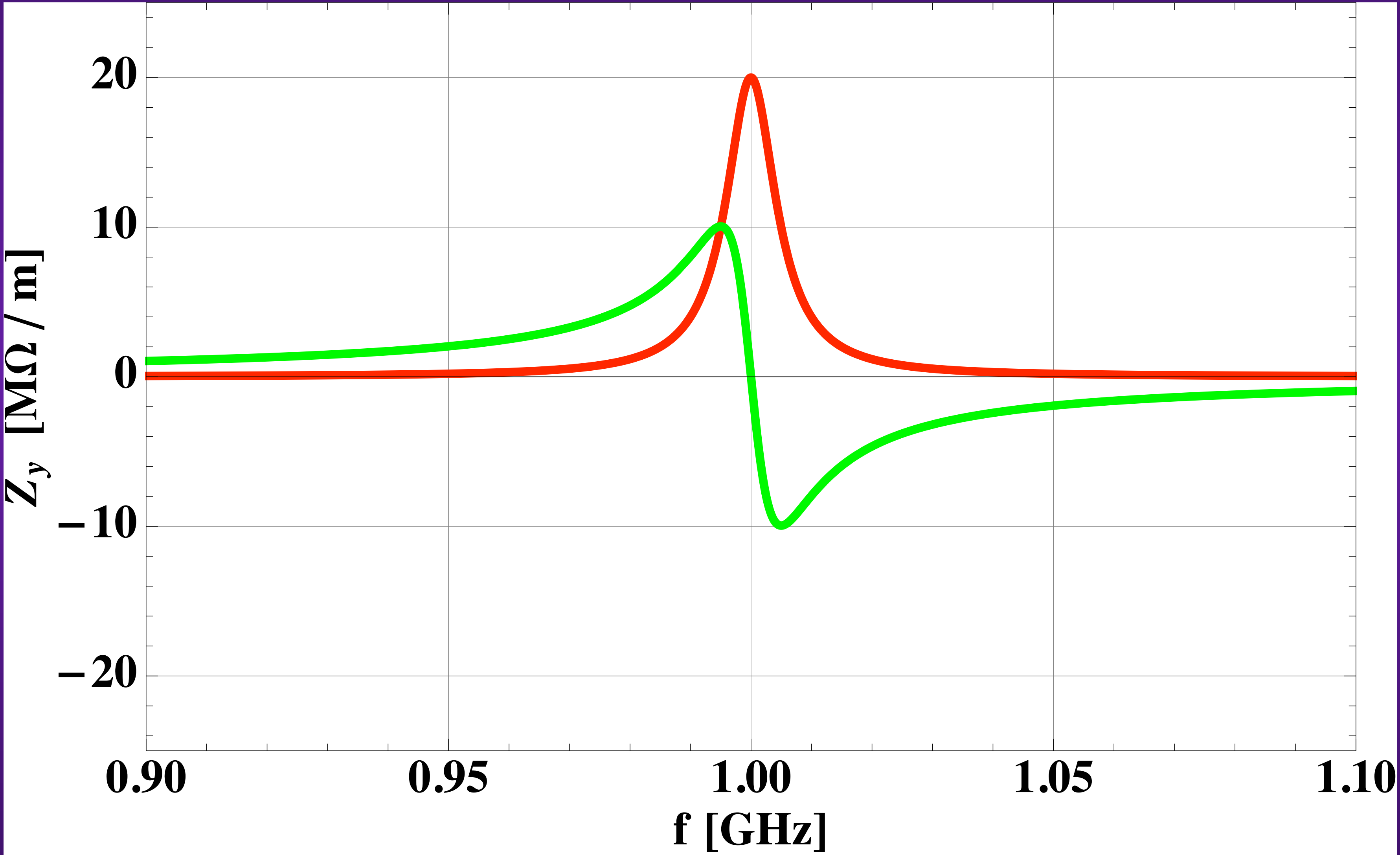
IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (22/25)

- **Example 3 in the transverse plane (resonator wake field)**



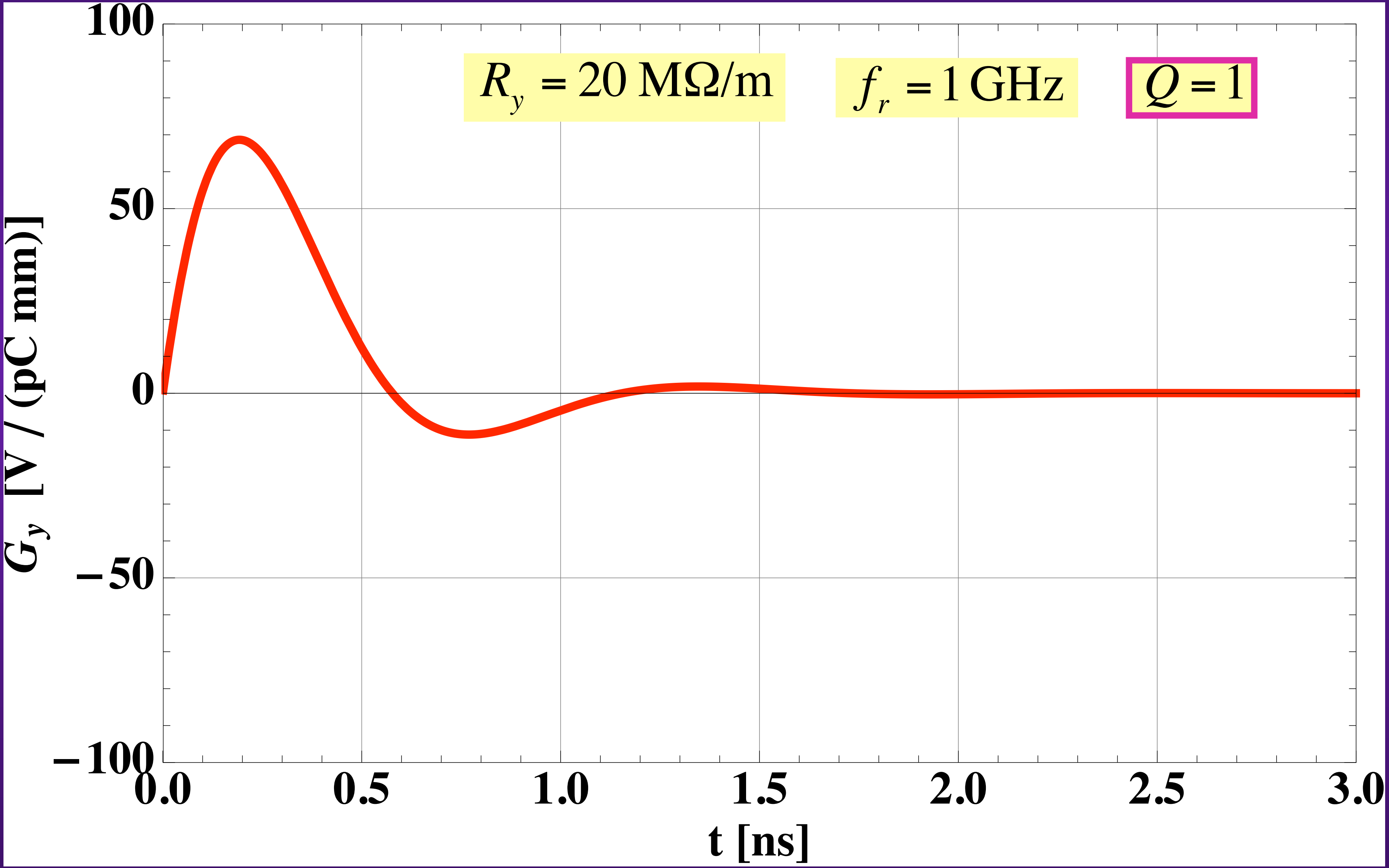
IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (23/25)

- **Example 3 in the transverse plane (resonator impedance)**



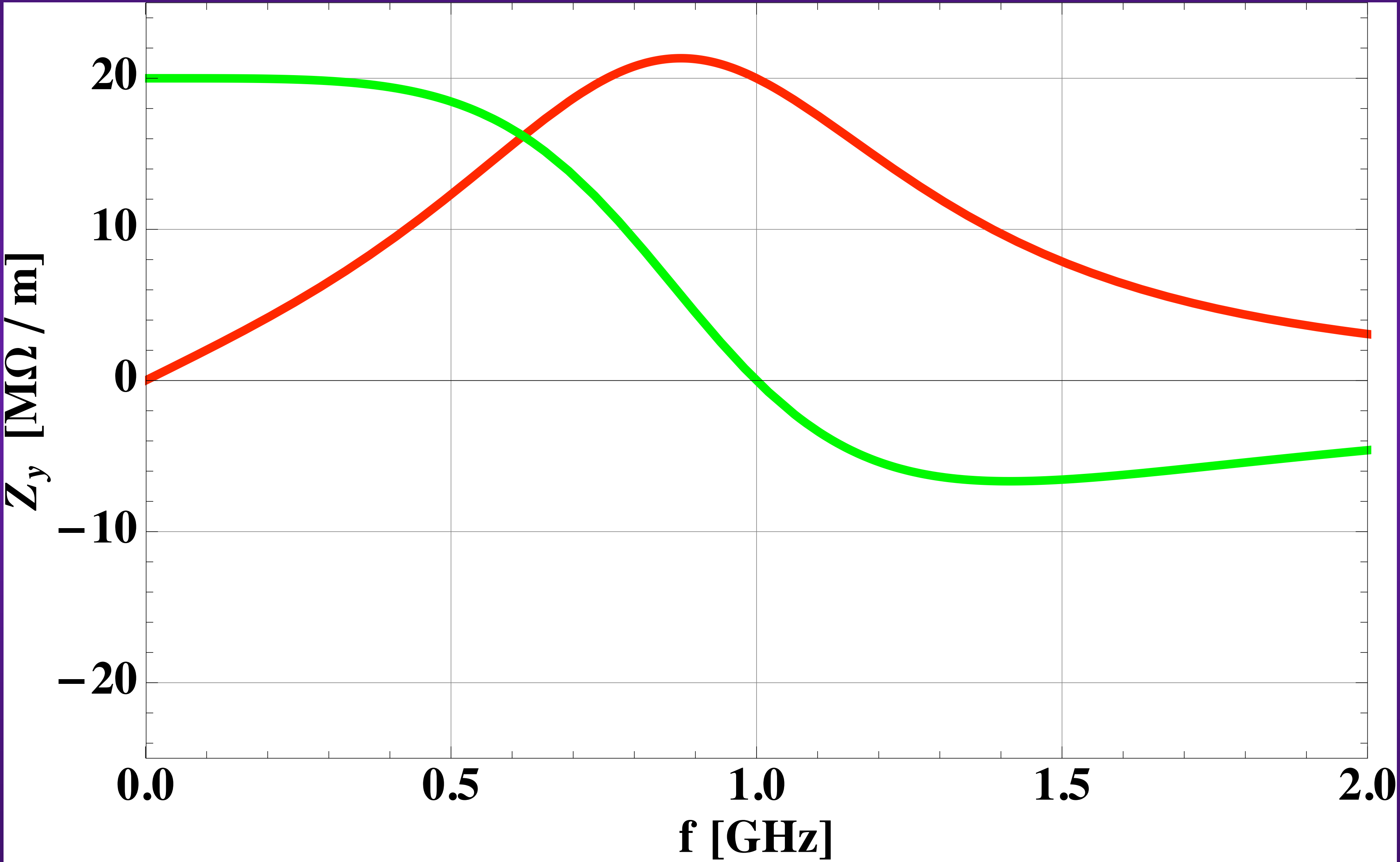
IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (24/25)

- Example 4 in the transverse plane (resonator wake field)



IMPEDANCE AND WAKE POTENTIAL OF A RESONATOR (25/25)

- **Example 4 in the transverse plane (resonator impedance)**



CUT-OFF FREQUENCIES IN A CIRCULAR WAVEGUIDE (1/7)

- **One distinguishes between (in vacuum)**
  - **TM (Transverse Magnetic) modes**
  - **TE (Transverse Electric) modes**

See previous slides

- **TM**

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial s^2}+\omega^2\mu\,\varepsilon_c\right]E_s=0 \qquad B_s=0$$

=>

$$\boxed{\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\left(\frac{\omega}{c}\right)^2-k^2-\frac{m^2}{r^2}\right]R(r)=0}$$

with $E_s=e^{j\omega t}\,\Theta(\theta)\,S(s)\,R(r)$ $\qquad \Theta(\theta)=e^{\pm jm\theta}$ $\qquad S(s)=e^{\pm jks}$

CUT-OFF FREQUENCIES IN A CIRCULAR WAVEGUIDE (2/7)

The solution of this differential equation is the mth Bessel function

$$R(r) = J_m(k_r\, r)$$

with $$k_r^2 = \left(\frac{\omega}{c}\right)^2 - k^2$$

Radial wave number

=> $$E_s = E_{sm0}\, J_m(k_r\, r)\, e^{jm\theta}\, e^{j(\omega t - k s)}$$

See (before) general relations between longitudinal and transverse components

and $E_\theta = 0$ **if** $E_s = 0$

CUT-OFF FREQUENCIES IN A CIRCULAR WAVEGUIDE (3/7)

The propagation modes are determined by the boundary condition for $\boxed{E_s = E_\theta = 0}$ at the pipe radius $r = b$

$$\boxed{k_{r,\,mn} = \frac{j_{mn}}{b}}$$

where j_{mn} is the nth zero of the mth Bessel function

=> The frequency of the mode TM$_{mn}$ is given by $\left(\frac{\omega}{c}\right)^2 = \left(\frac{j_{mn}}{b}\right)^2 + k^2$

=> $$f = \frac{c}{2\pi}\sqrt{\left(\frac{j_{mn}}{b}\right)^2 + k^2}$$

CUT-OFF FREQUENCIES IN A CIRCULAR WAVEGUIDE (4/7)

The cut-off frequency of the TM_{mn} mode is defined by

$$\boxed{f_{TM_{mn}}^{cut-off} = \frac{c}{b} \times \frac{j_{mn}}{2\pi}}$$

Below this frequency propagation is not possible as in this case $k^2 < 0$ and therefore k is not real

The lowest cut-off frequency is given by the 1st zero of the Bessel function of 0th order, which is

$$\boxed{j_{01} \approx 2.4}$$

=> $$\boxed{f_{TM_{01}}^{cut-off} = \frac{c}{b} \times \frac{2.4}{2\pi} \approx 0.4 \, \frac{c}{b}}$$

CUT-OFF FREQUENCIES IN A CIRCULAR WAVEGUIDE (5/7)

- **TE**

$$\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial s^2}+\omega^2\mu\,\varepsilon_c\right]B_s=0 \qquad \boxed{E_s=0}$$

A similar analysis as before can be performed, leading to

$$B_s=B_{sm0}\,J_m\left(k_r\,r\right)e^{\,j\,m\,\theta}\;e^{\,j\left(\omega\,t-k\,s\right)}$$

However, in this case the boundary condition (at the pipe radius $r=b$) is $B_r=0$, which is equivalent to (looking at the relations between longitudinal and transverse components)

$$\boxed{\frac{d\,B_s}{d\,r}=0}$$

CUT-OFF FREQUENCIES IN A CIRCULAR WAVEGUIDE (6/7)

=> $$\boxed{k_{r\,,\,mn} = \frac{j'_{mn}}{b}}$$

where j'_{mn} is the nth zero of the derivative of the mth Bessel function

=> The frequency of the mode TE$_{mn}$ is given by $$\left(\frac{\omega}{c}\right)^2 = \left(\frac{j'_{mn}}{b}\right)^2 + k^2$$

=> $$f = \frac{c}{2\pi}\sqrt{\left(\frac{j'_{mn}}{b}\right)^2 + k^2}$$

CUT-OFF FREQUENCIES IN A CIRCULAR WAVEGUIDE (7/7)

The cut-off frequency of the TE_{mn} mode is defined by

$$\boxed{f_{TE_{mn}}^{cut-off} = \frac{c}{b} \times \frac{j'_{mn}}{2\pi}}$$

Below this frequency propagation is not possible as in this case $k^2 < 0$ and therefore k is not real

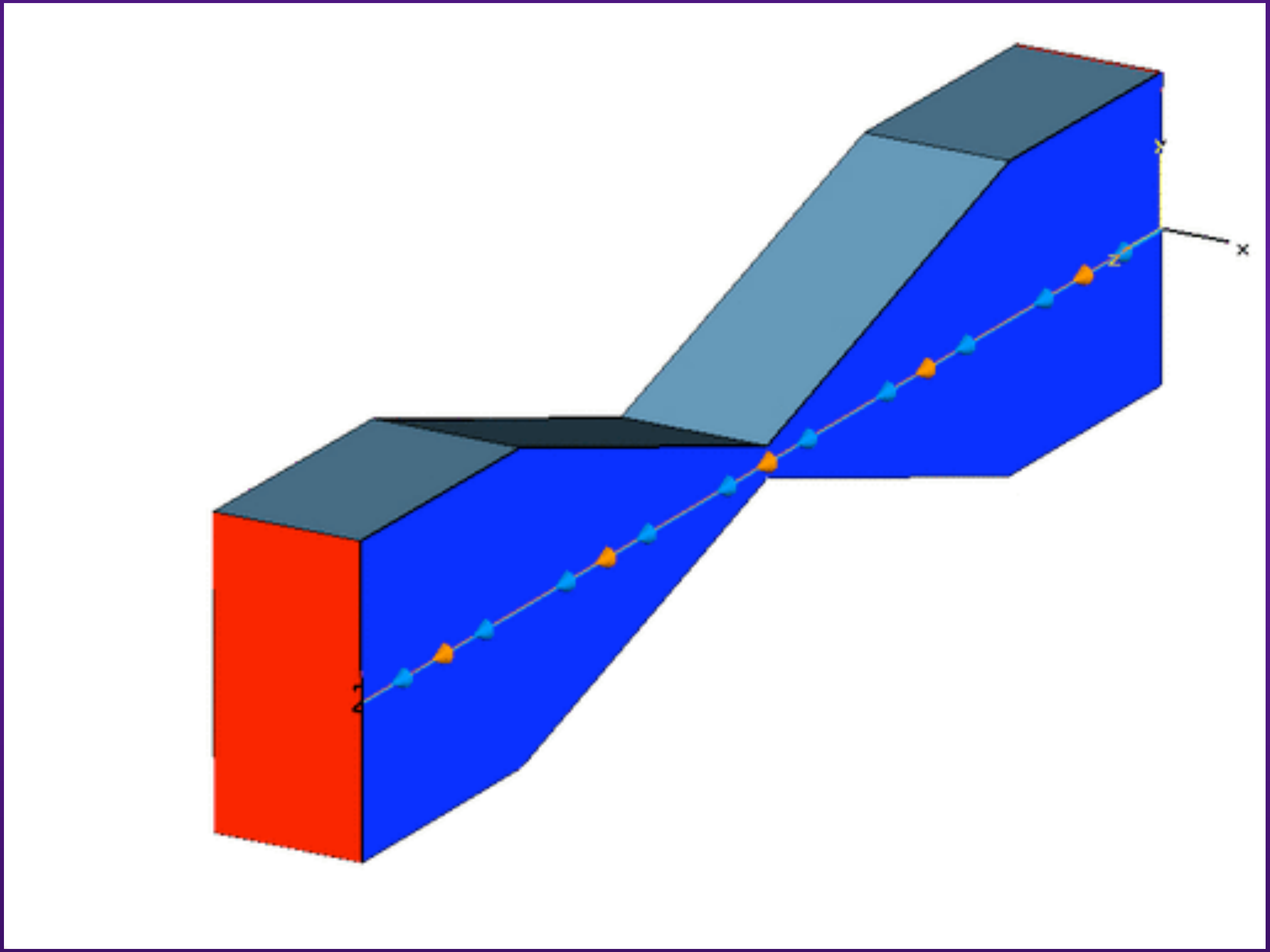
The lowest cut-off frequency is given by the 1st zero of the derivative of the Bessel function of 1th order, which is

$$\boxed{j'_{11} \approx 1.84}$$

=> $$\boxed{f_{TE_{11}}^{cut-off} = \frac{c}{b} \times \frac{1.84}{2\pi} \approx 0.3\,\frac{c}{b}}$$

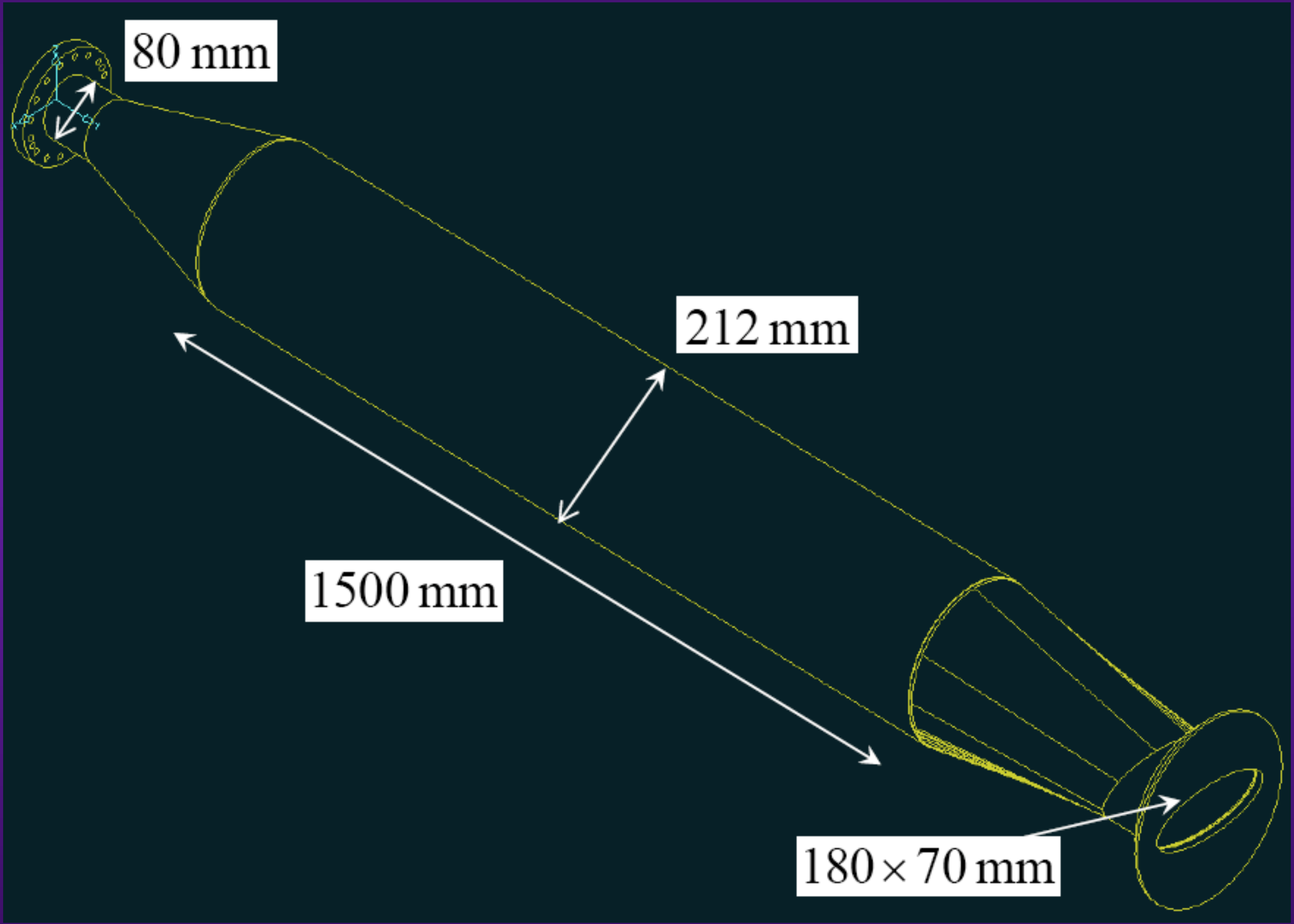
EXAMPLES OF ELECTROMAGNETIC SIMULATIONS (1/17)

- **Example from CST (Computer Simulation Technology: http://www.cst.com/Content/Applications/Article/Wake+Field+Simulation+of+a+Collimator) => Wake field simulation of a collimator**



EXAMPLES OF ELECTROMAGNETIC SIMULATIONS (2/17)

- **A tertiary LHC collimator chamber with the HFSS code**



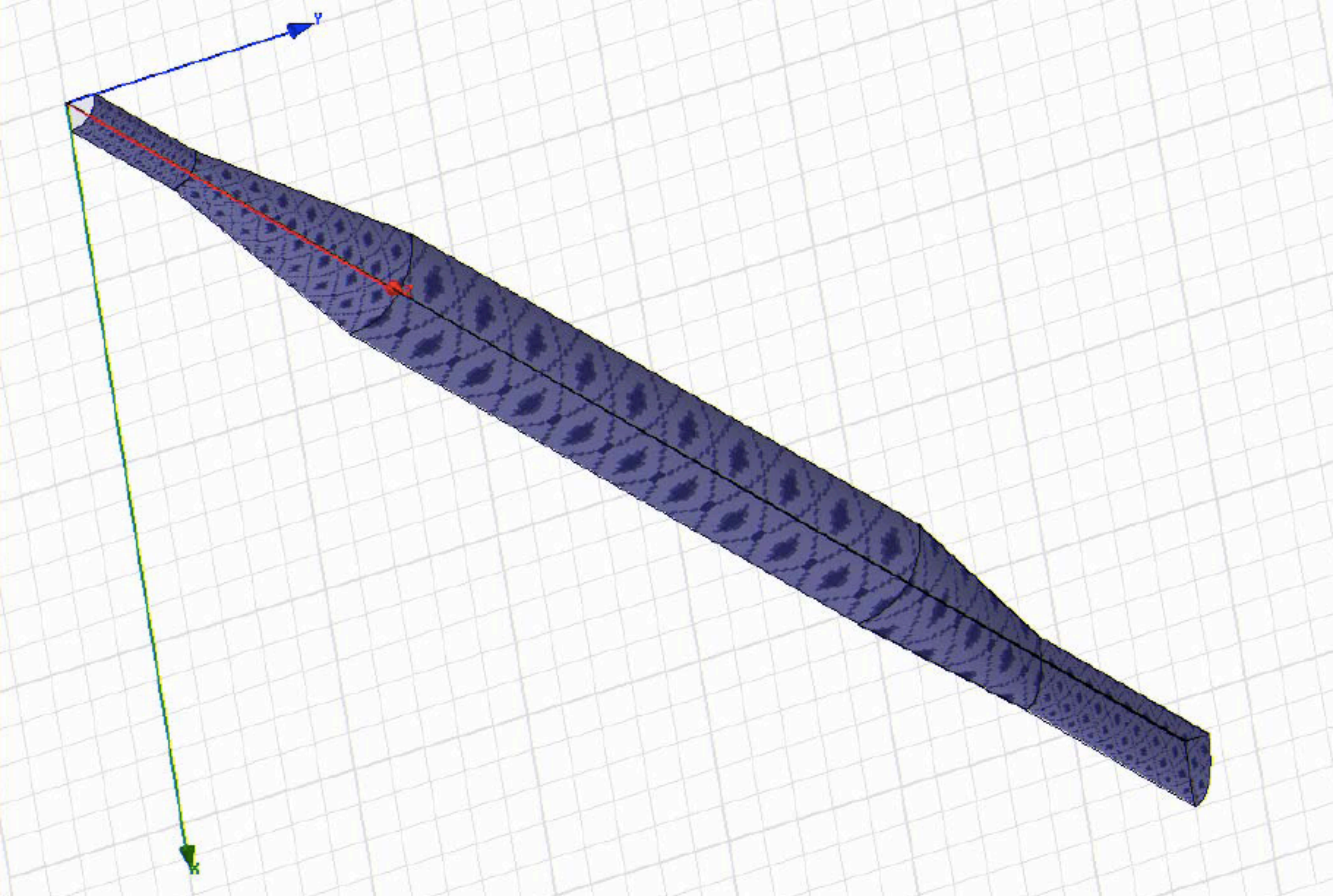
EXAMPLES OF ELECTROMAGNETIC SIMULATIONS (3/17)

=> One can already anticipate some resonances (trapped modes) above the lowest (i.e. of the largest beam pipe radius *b*) cut-off frequency

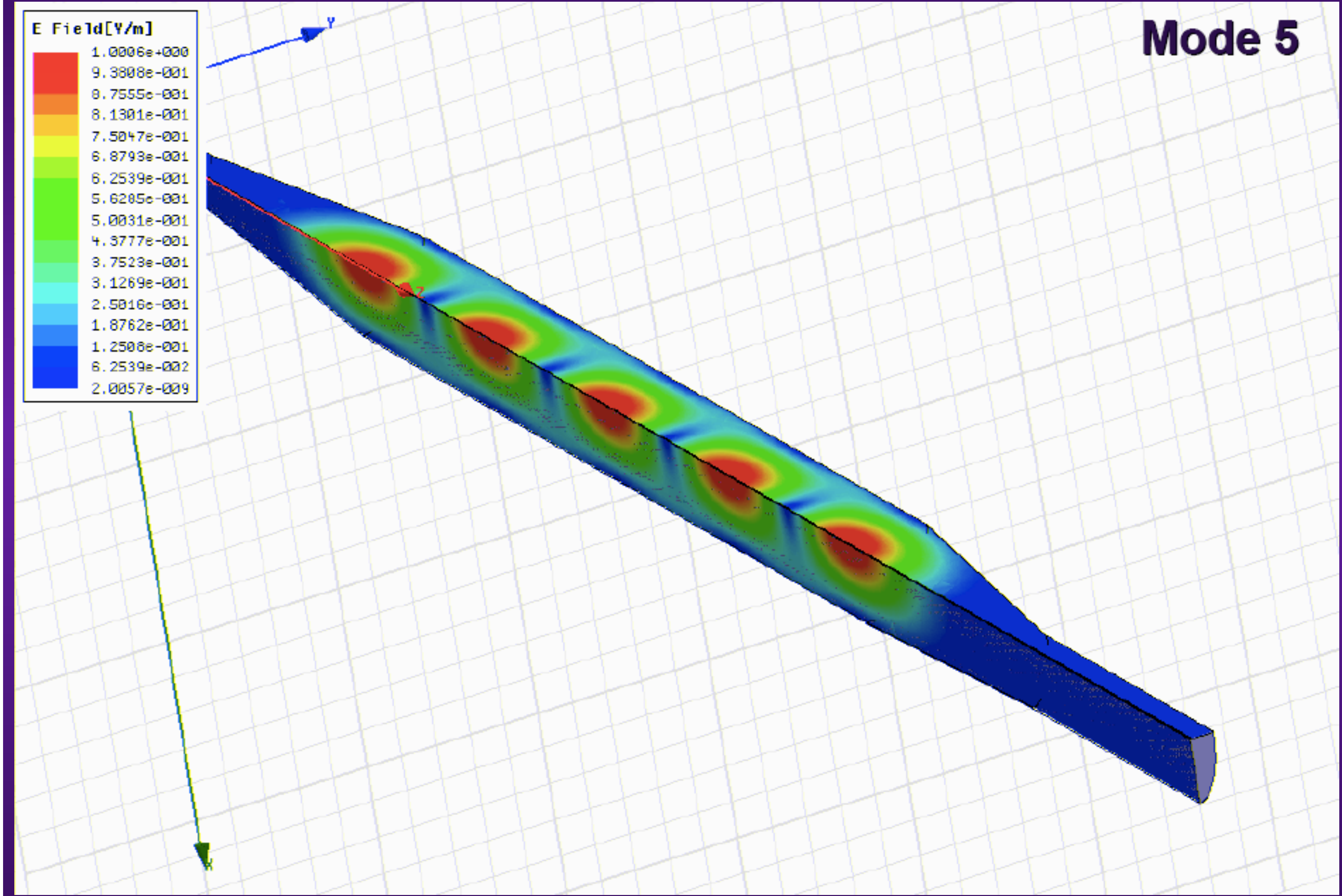
$$\boxed{f_{cut-off}^{\,lowest}\ [\mathrm{GHz}] \approx \frac{10}{b\,[\mathrm{cm}]}}$$

As $b_{\mathrm{largest}} = \dfrac{212}{2}\ \mathrm{mm} = 10.6\ \mathrm{cm}$, the first resonance should be around 1 GHz

EXAMPLES OF ELECTROMAGNETIC SIMULATIONS (4/17)



EXAMPLES OF ELECTROMAGNETIC SIMULATIONS (5/17)



EXAMPLES OF ELECTROMAGNETIC SIMULATIONS (6/17)

GHz		Ω
$f_{r1} = 1.0857$	$Q_1 = 7113.9$	$R_1 = 12.1$
$f_{r2} = 1.0948$	$Q_2 = 7120.3$	$R_2 = 75.2$
$f_{r3} = 1.1098$	$Q_3 = 7135.9$	$R_3 = 18.1$
$f_{r4} = 1.1305$	$Q_4 = 7158.7$	$R_4 = 302.6$
$f_{r5} = 1.1565$	$Q_5 = 7194.1$	$R_5 = 158.3$
$f_{r6} = 1.1872$	$Q_6 = 7374.7$	$R_6 = 74.8$
$f_{r7} = 1.2218$	$Q_7 = 8914.4$	$R_7 = 555.6$
$f_{r8} = 1.2596$	$Q_8 = 4488.6$	$R_8 = 143.1$
$f_{r9} = 1.3000$	$Q_9 = 1743.3$	$R_9 = 3.8$
$f_{r10} = 1.3474$	$Q_{10} = 2220.0$	$R_{10} = 116.0$

EXAMPLES OF ELECTROMAGNETIC SIMULATIONS (7/17)

Power loss for mode i

See GR's talk

$$P_{loss}^{\mathrm{Max}} = 2\left(M\, I_b\right)^2 R_i\, e^{-\left(\omega_{ri}\,\sigma_\tau\right)^2}$$

$M = 3564$

$I_b = N_b\, e\, f_0$

$N_b = 1.15 \times 10^{11}$ p/b

$f_0 = 11245.5$ Hz

$\sigma_\tau = 0.25$ ns

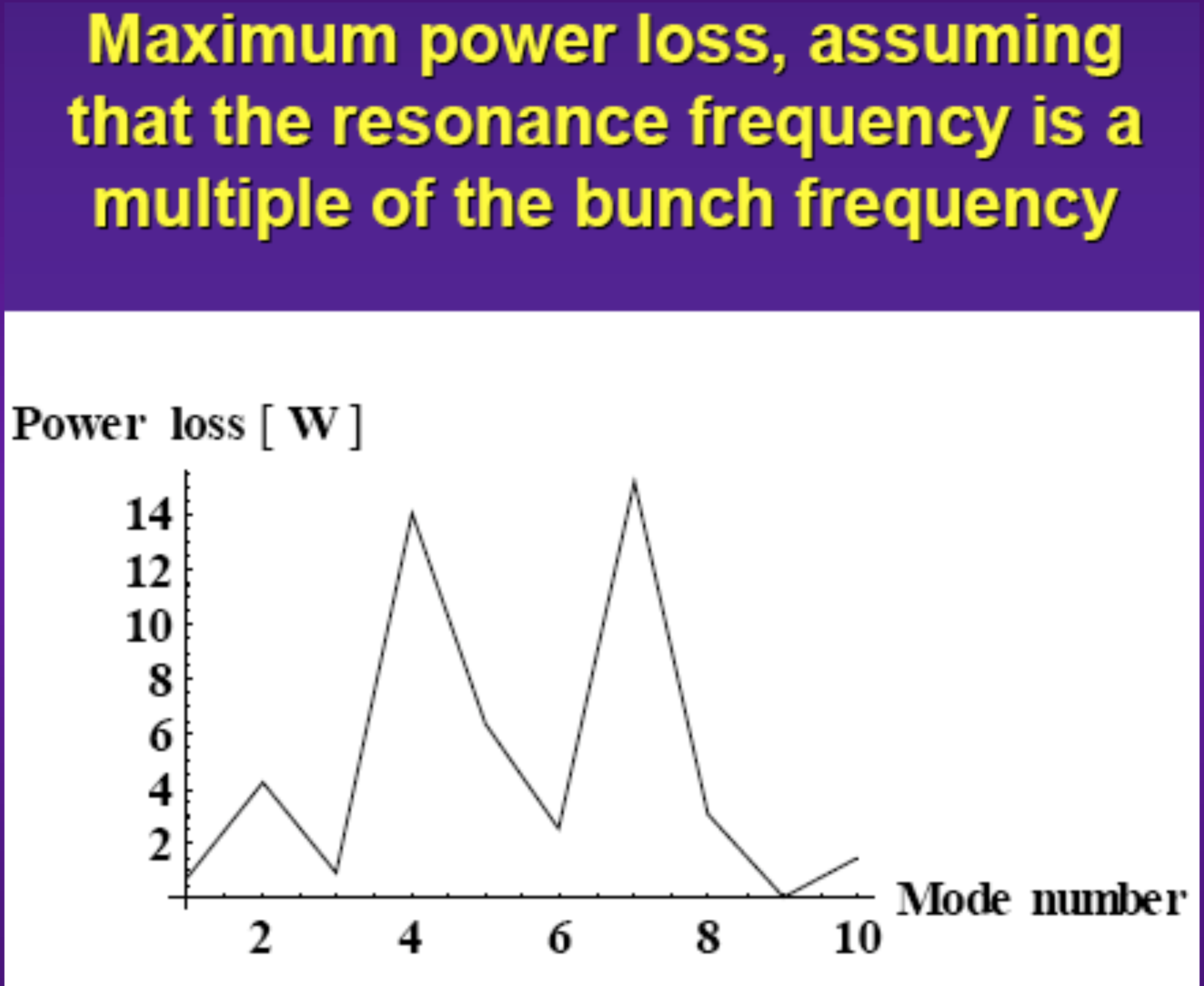
$f_b = M\, f_0 = 40$ MHz

and

$$P_{loss}^{\mathrm{"Real"}} = P_{loss}^{\mathrm{Max}} \times \frac{\Delta^2}{\Delta^2 + \sin^2\left(\dfrac{\pi\, f_{ri}}{f_b}\right)}$$

if $\Delta = \dfrac{\pi\, f_{ri}}{2\, Q_i\, f_b} << 1$ $Q_i >> 1$

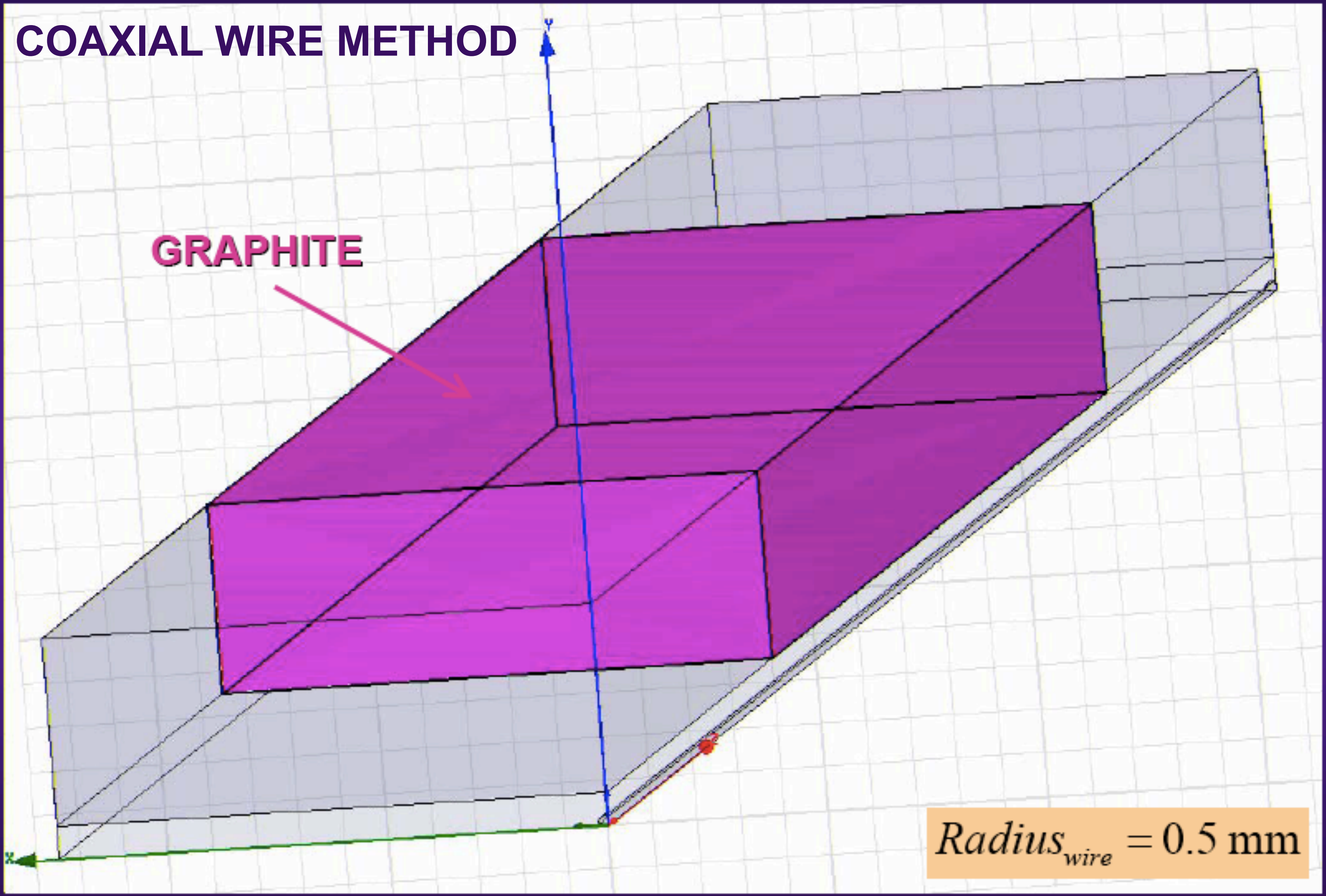
EXAMPLES OF ELECTROMAGNETIC SIMULATIONS (8/17)



Maximum power loss, assuming that the resonance frequency is a multiple of the bunch frequency

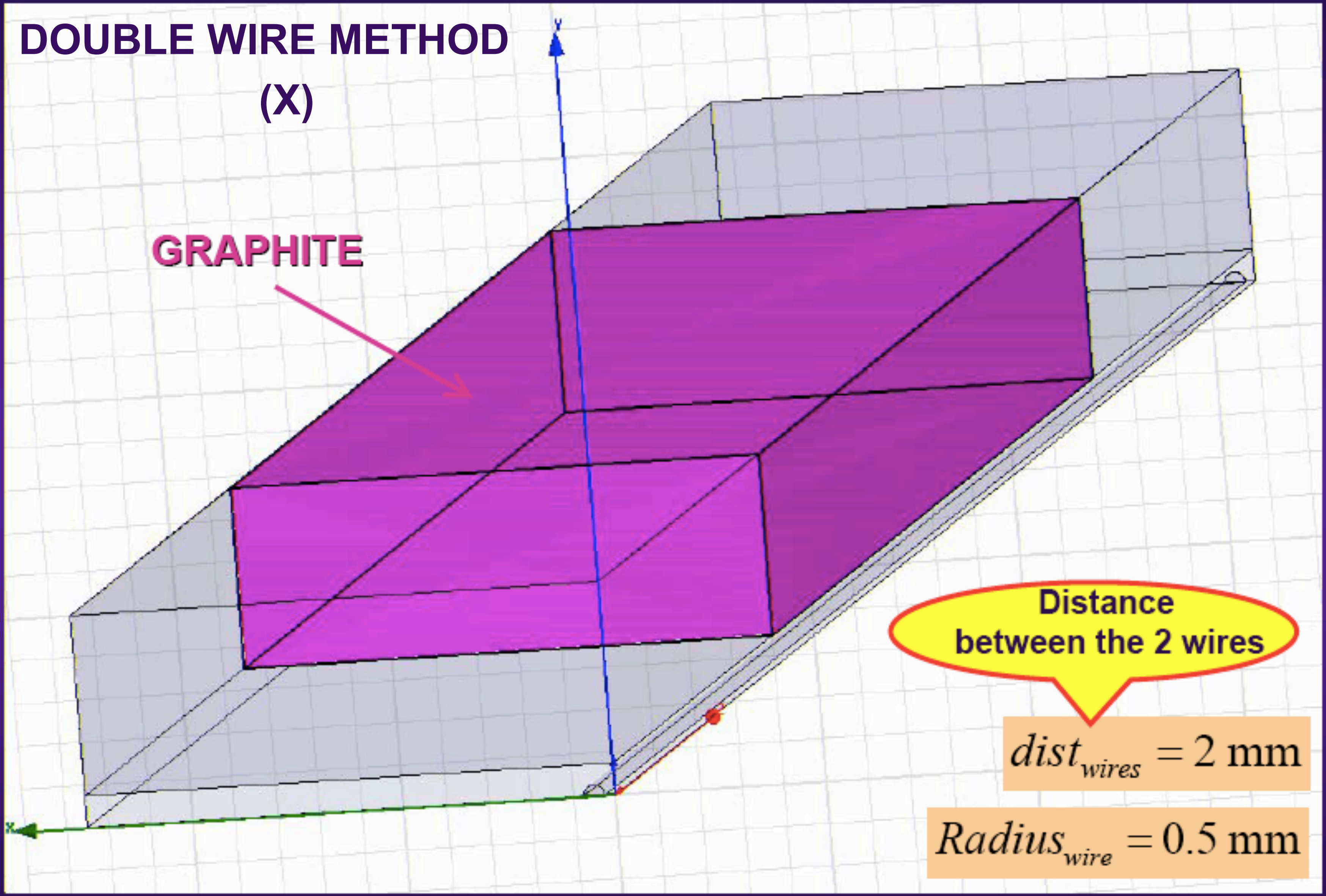
EXAMPLES OF ELECTROMAGNETIC SIMULATIONS (9/17)

- **A LHC graphite collimator with the HFSS code**



$Radius_{wire} = 0.5\ \mathrm{mm}$

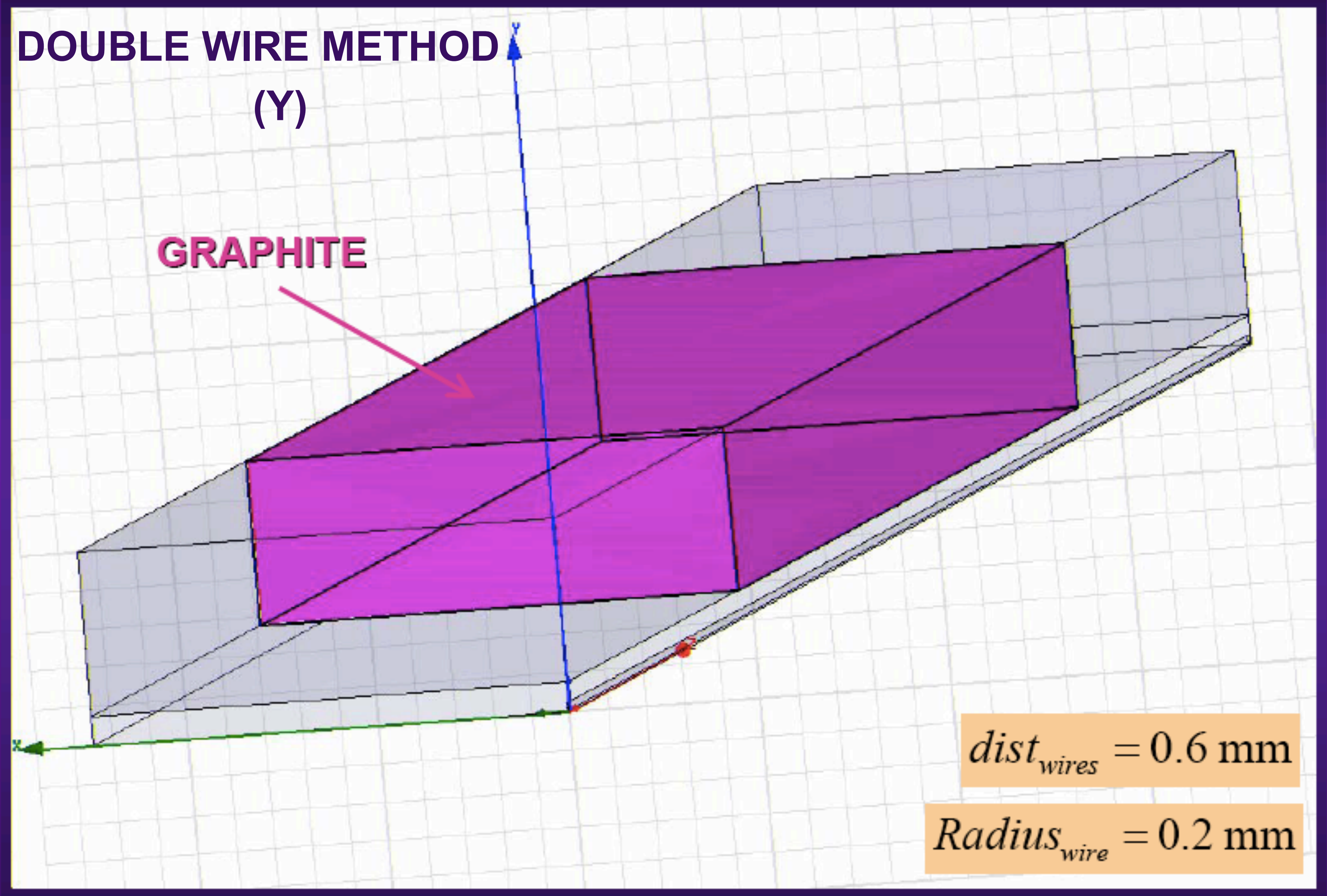
EXAMPLES OF ELECTROMAGNETIC SIMULATIONS (10/17)



$dist_{wires} = 2 \text{ mm}$

$Radius_{wire} = 0.5 \text{ mm}$

EXAMPLES OF ELECTROMAGNETIC SIMULATIONS (11/17)



DOUBLE WIRE METHOD (Y)

$dist_{wires} = 0.6\text{ mm}$

$Radius_{wire} = 0.2\text{ mm}$

EXAMPLES OF ELECTROMAGNETIC SIMULATIONS (12/17)

Improved log formula for distributed (i.e. not lumped) impedances

- **Longitudinal**

$$Z_l = -2\, Z_{ch} \log\left(\frac{S_{21}}{S_{REF}}\right)$$

- S$_{21}$ is deduced from HFSS
- $Z_{ch} = 60 \log\left(1.27 \frac{b_1}{Radius_{wire}}\right)$
- $S_{REF} = e^{-j\,\omega \frac{L}{c}}$

Also computed with HFSS by Tsutsui

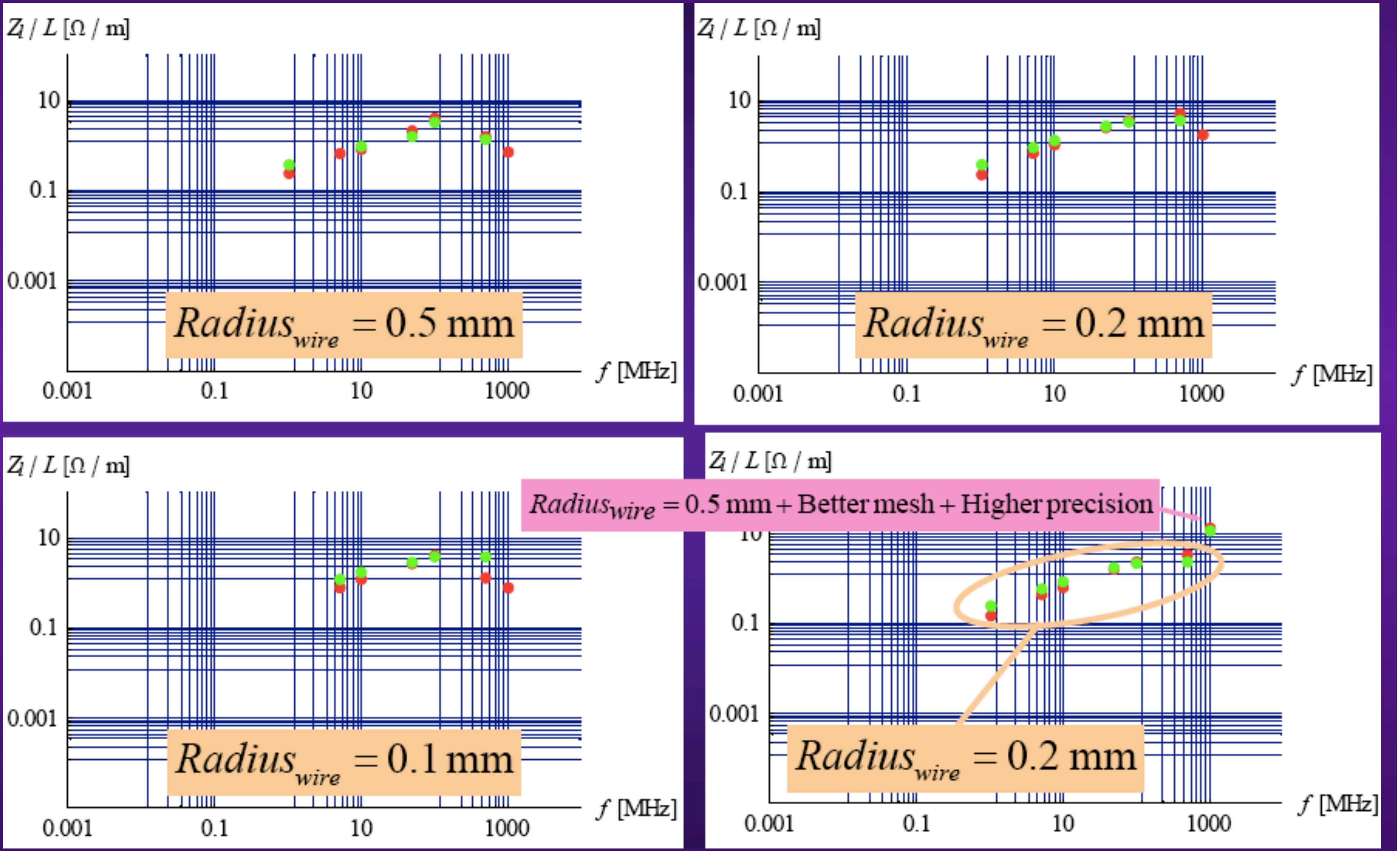
- **Transverse**

$$Z_\perp = -2\, Z_{ch} \frac{c}{\omega\, dist_{wires}^2} \log\left(\frac{S_{21}}{S_{REF}}\right)$$

- S$_{21}$ is deduced from HFSS
- $Z_{ch} = 120\, \mathrm{ArcCosh}\left(\frac{dist_{wires}}{2\, Radius_{wire}}\right)$
- $S_{REF} = e^{-j\,\omega \frac{L}{c}}$

EXAMPLES OF ELECTROMAGNETIC SIMULATIONS (13/17)

Longitudinal impedance => Real (imaginary) part in red (green)



$Radius_{wire} = 0.5\text{ mm}$

$Radius_{wire} = 0.2\text{ mm}$

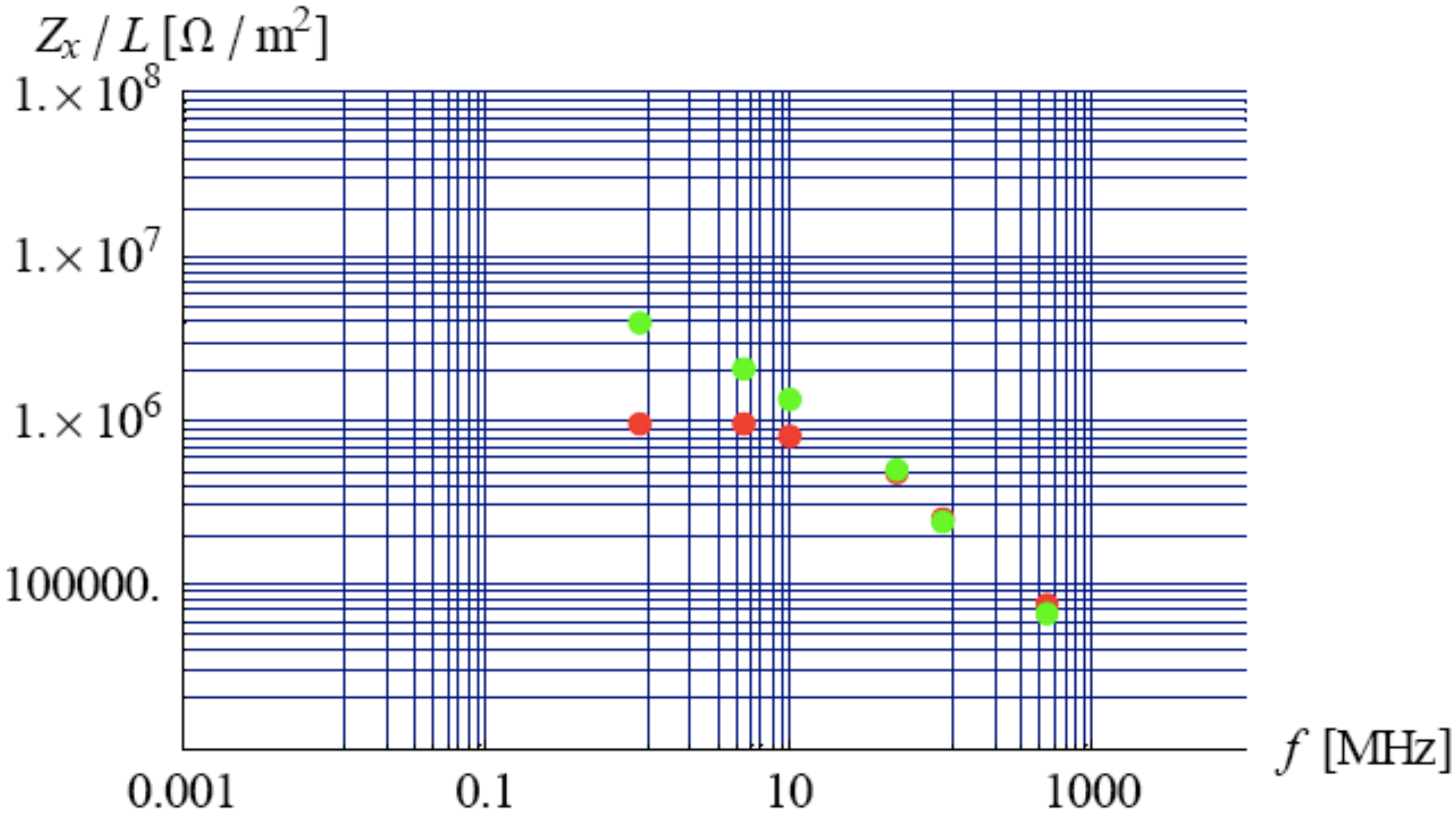
$Radius_{wire} = 0.1\text{ mm}$

$Radius_{wire} = 0.5\text{ mm} + \text{Better mesh} + \text{Higher precision}$

$Radius_{wire} = 0.2\text{ mm}$

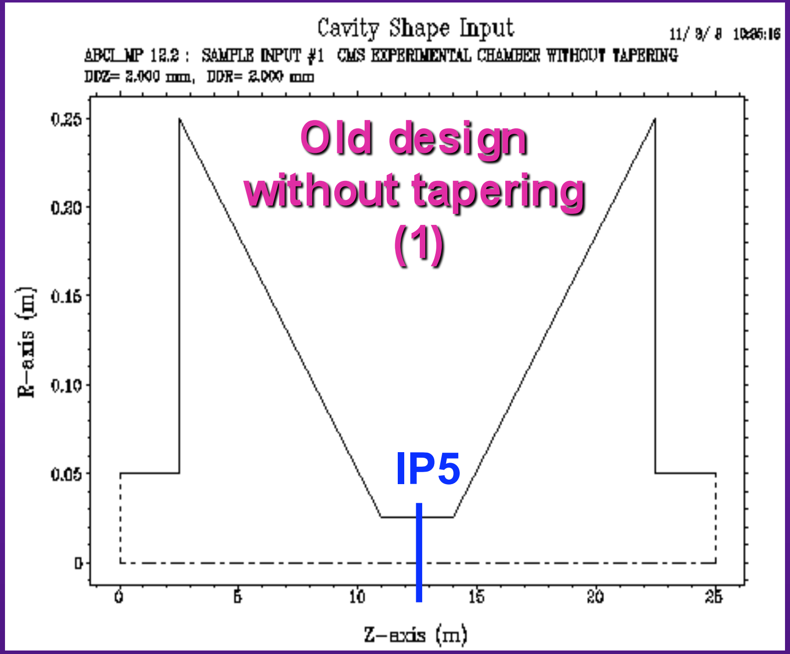
EXAMPLES OF ELECTROMAGNETIC SIMULATIONS (14/17)

Horizontal impedance



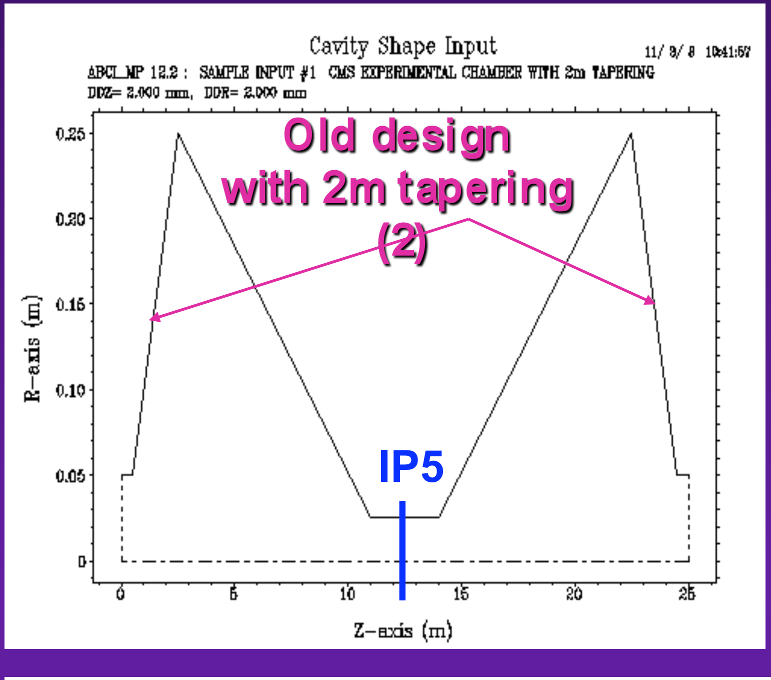
EXAMPLES OF ELECTROMAGNETIC SIMULATIONS (15/17)

- **The CMS vacuum chamber (in the LHC) with ABCI code**



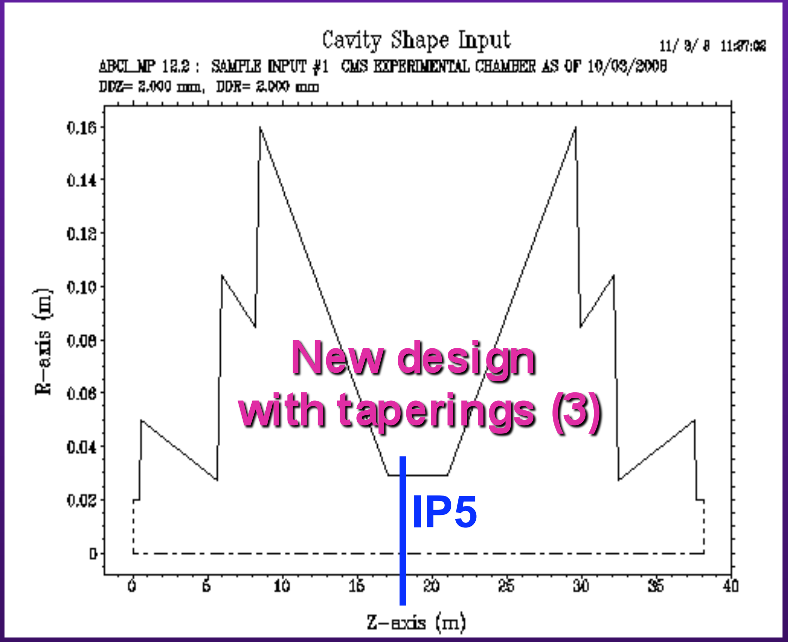
Old design without tapering (1)

IP5



Old design with 2m tapering (2)

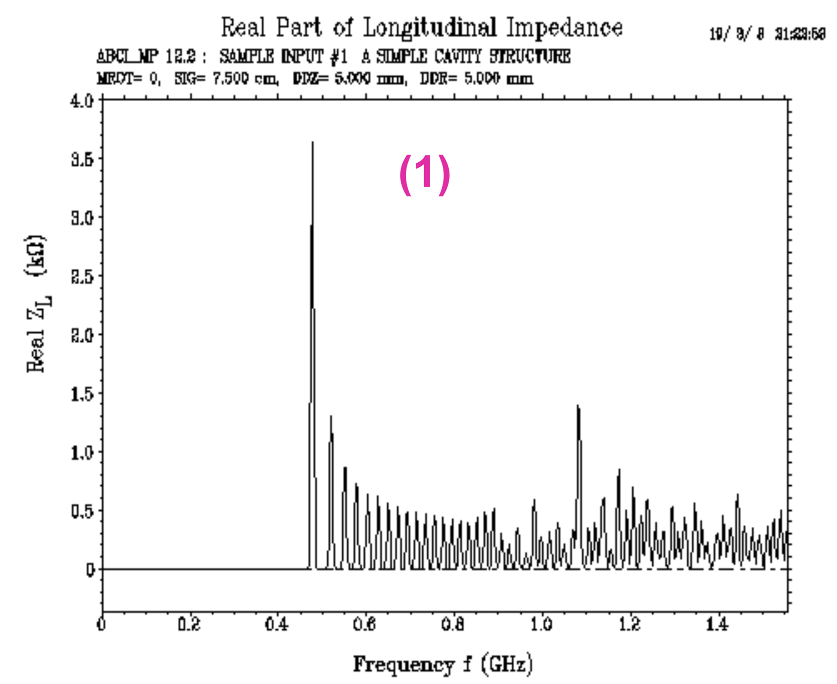
IP5

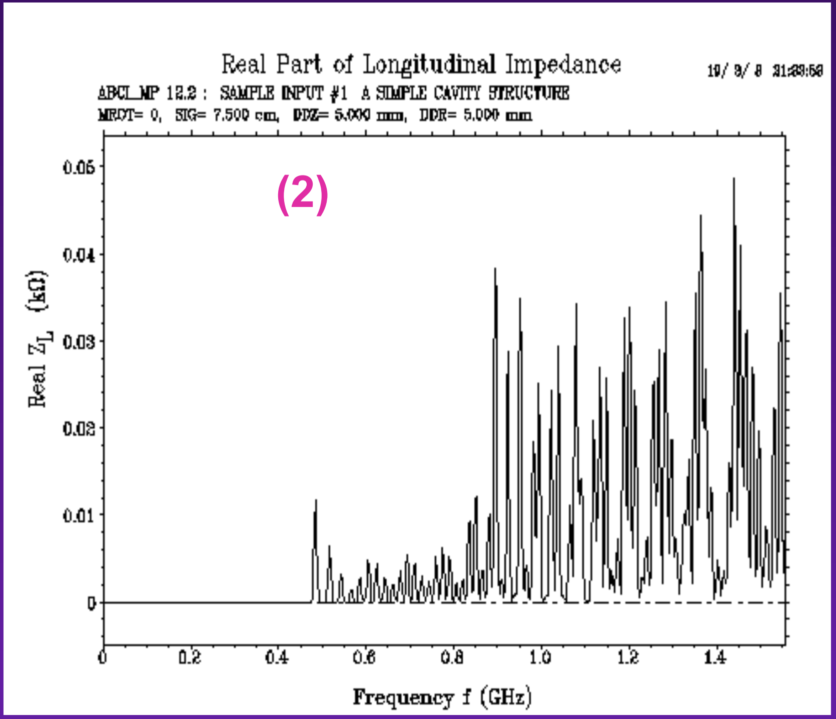


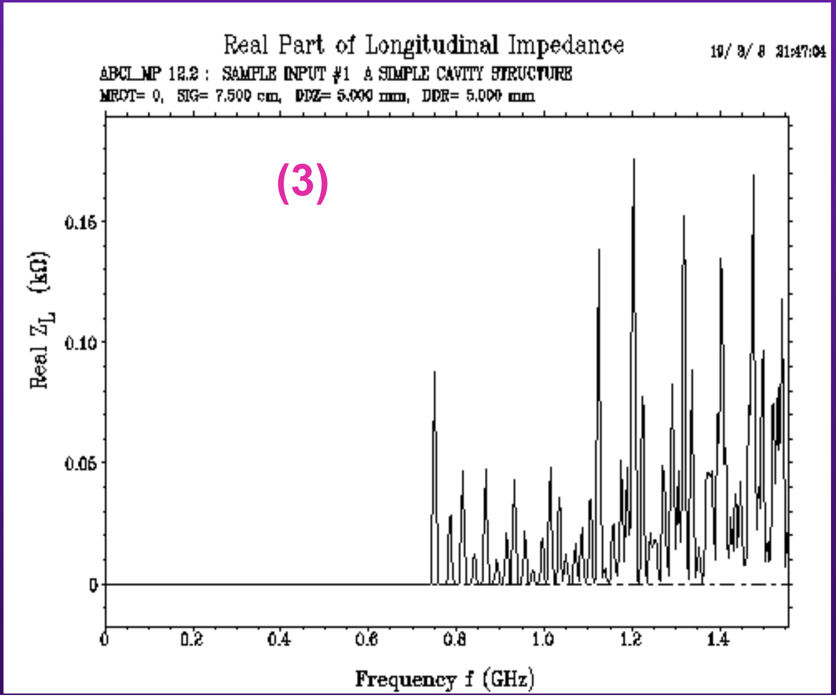
New design with taperings (3)

IP5

EXAMPLES OF ELECTROMAGNETIC SIMULATIONS (16/17)



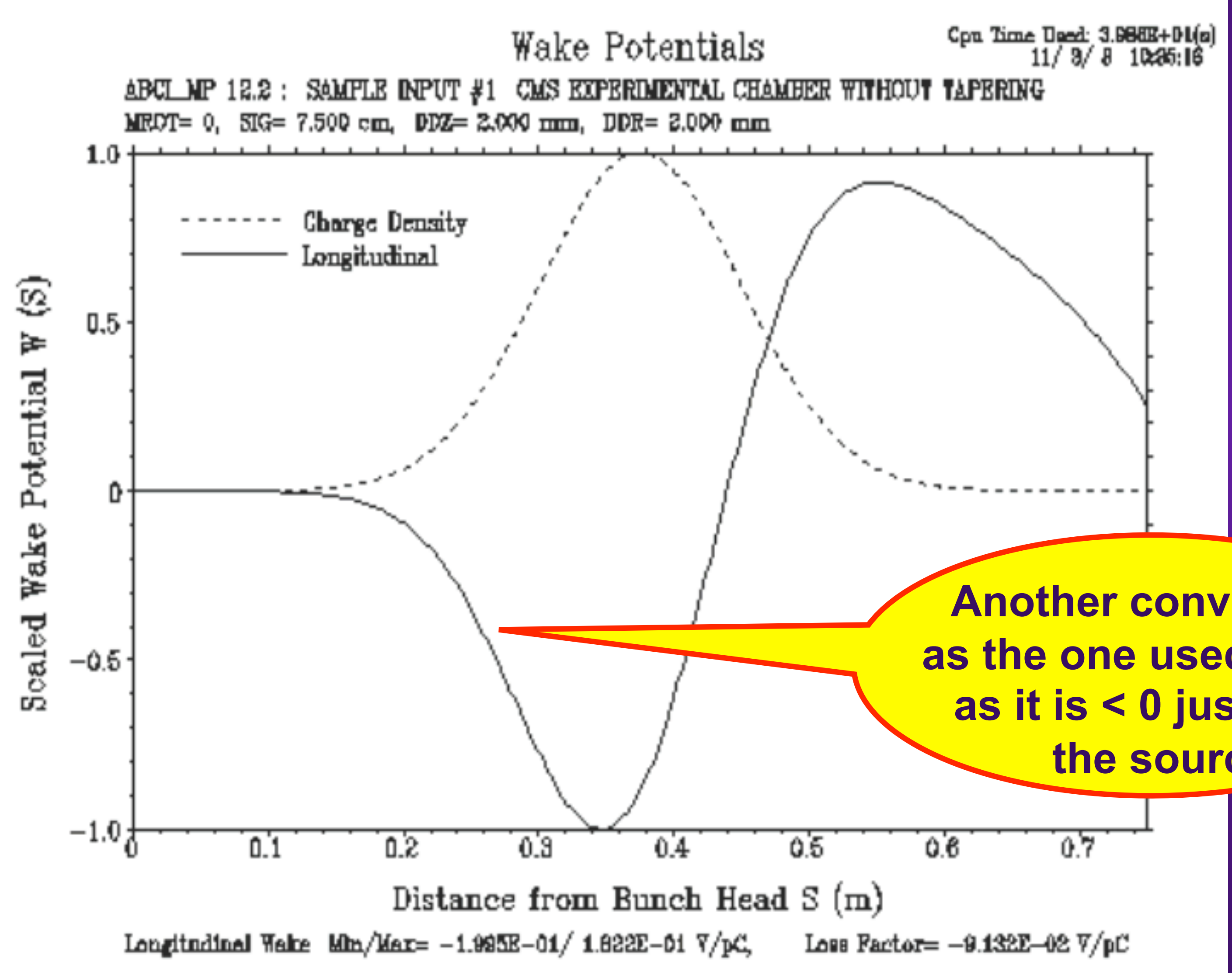




=> The resonance frequency of the 1st mode is shifted as expected from something between 450 and 500 MHz to ~ 750 MHz (when the larger beam pipe radius reduces from 25 cm to 16 cm)

EXAMPLES OF ELECTROMAGNETIC SIMULATIONS (17/17)

Example of simulated longitudinal wake potential for the case of the old design without tapering



Another convention as the one used before as it is < 0 just after the source