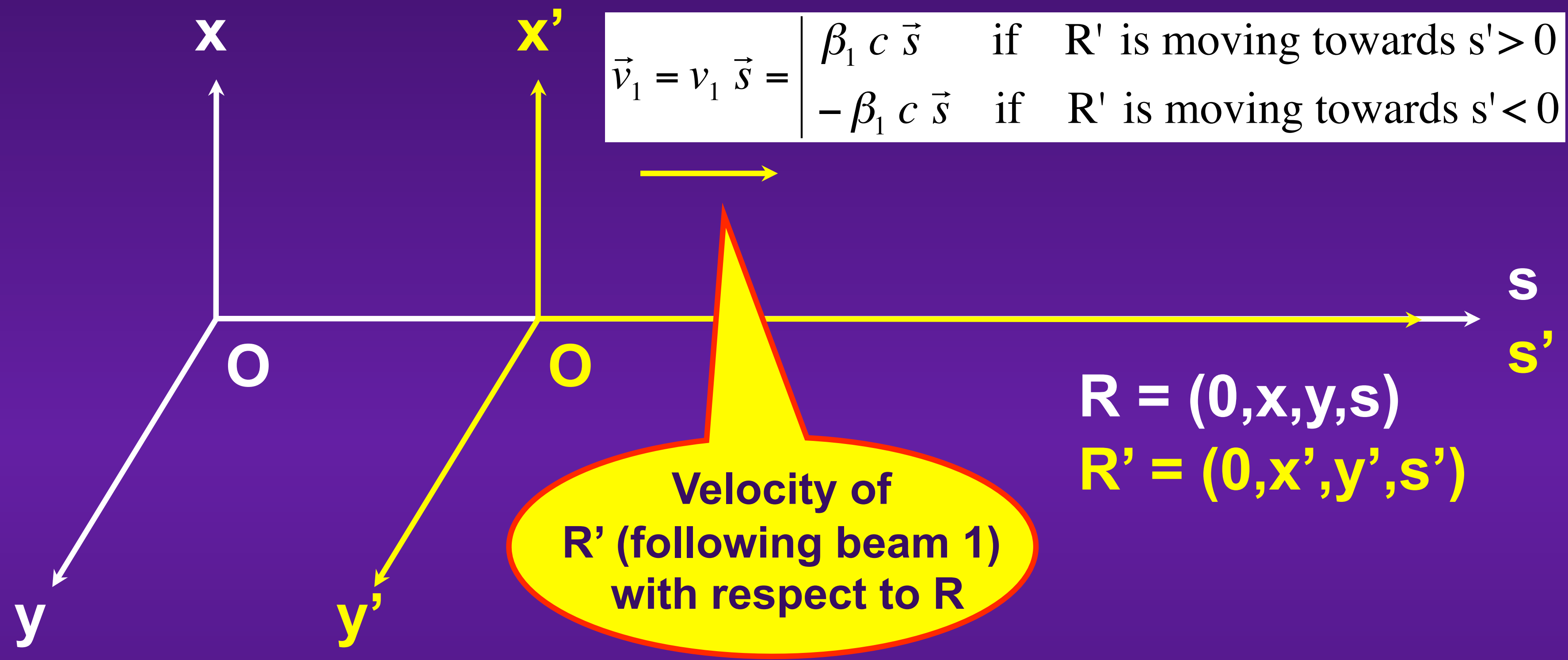
SPACE CHARGE

- Reminder: Relativistic transformation of the EM fields (1 slide)
- Lorentz force (3)
- Panofsky-Wenzel theorem (2)
- EM fields of a cylinder with uniform density (3)
- EM fields for a bunch with Gaussian densities in r and s (3)
- Transverse and longitudinal incoherent tune shifts (10)
- Transverse (direct) tune spreads (6)
- Effect of the images (i.e. the wall)
  - Beam off-axis in a Perfectly Conducting circular beam pipe (5)
  - Beam off-axis between 2 infinite PC // plates (12)
- General formulae for the tune shifts of coasting&bunched beams (9)
- A practical formula for the maximum transverse incoherent direct SC tune shift (1)
- Transverse incoherent direct SC tune shift formula for ions (1)

REMINDER: RELATIVISTIC TRANSFORMATION OF THE EM FIELDS

$$\vec{v}_1 = v_1\,\vec{s} = \left|\begin{array}{lll} \beta_1\, c\, \vec{s} & \text{if} & \text{R' is moving towards s'} > 0 \\ -\beta_1\, c\, \vec{s} & \text{if} & \text{R' is moving towards s'} < 0 \end{array}\right.$$



$$E'_x = \gamma_1 \left(E_x - v_1\, B_y \right)$$

$$E'_y = \gamma_1 \left(E_y + v_1\, B_x \right)$$

$$E'_s = E_s \qquad B'_s = B_s$$

$$B'_x = \gamma_1 \left(B_x + \frac{v_1}{c^2}\, E_y \right)$$

$$B'_y = \gamma_1 \left(B_y - \frac{v_1}{c^2}\, E_x \right)$$

LORENTZ FORCE (1/3)

- **Lorentz force on the particle 2 moving with velocity** $\vec{v}_2 = v_2 \ \vec{s}$

$$\vec{F} = e \left(\vec{E} + \vec{v}_2 \times \vec{B} \right)$$

- **Beam 1 produces only an electric field in its rest frame R'**

$$B'_x = B'_y = B'_s = 0$$

=> $B_x = - \dfrac{v_1}{c^2} E_y$ $\quad B_y = \dfrac{v_1}{c^2} E_x$ $\quad B_s = 0$

=>

$$F_{x,y} = e \, E_{x,y} \left| \begin{array}{ll} \left(1 - \beta_1 \, \beta_2 \right) & \text{if 2 moves in same direction as 1} \\ \left(1 + \beta_1 \, \beta_2 \right) & \text{if 2 moves in oppo. direction as 1} \end{array} \right.$$

Space charge

Beam beam

LORENTZ FORCE (2/3)

- **Let's assume SC regime and $\beta_1 = \beta_2 = \beta$**

=> $$\boxed{F_{x,y} = e\, E_{x,y} \left(1 - \beta^2\right) = e\, \frac{E_{x,y}}{\gamma^2}}$$

Electric part

Magnetic part

and

$$E'_{x,y} = \frac{E_{x,y}}{\gamma}$$

$$B_x = -\frac{\beta}{c} E_y$$

$$E'_s = E_s$$

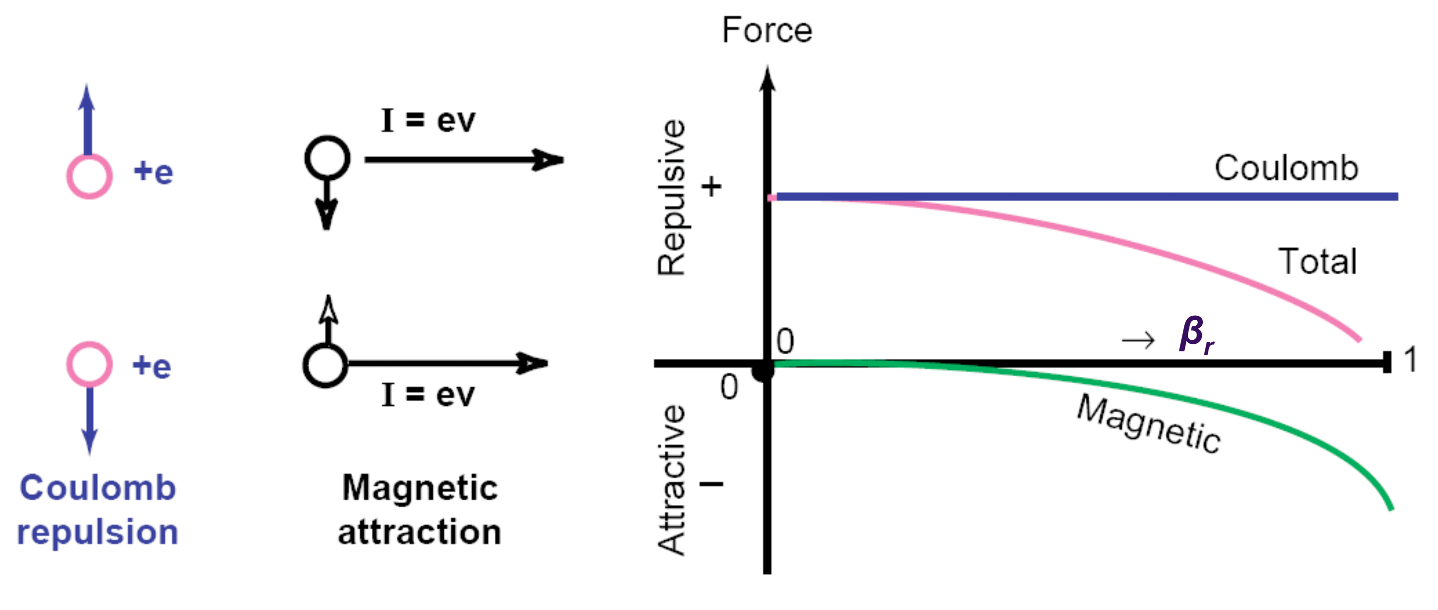
$$B_y = \frac{\beta}{c} E_x$$

$$B'_x = B'_y = B'_s = 0$$

$$B_s = 0$$

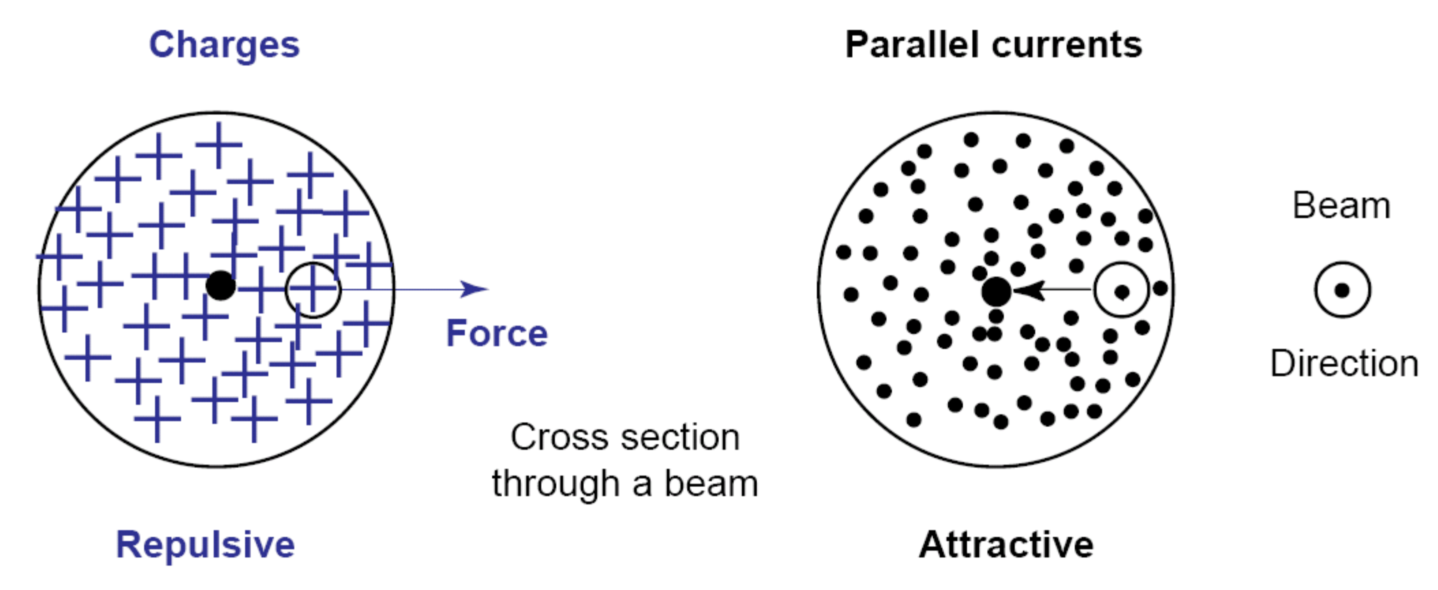
LORENTZ FORCE (3/3)

2 particles at rest or travelling



Courtesy K. Schindl

Many charged particles travelling in an unbunched beam with circular cross-section



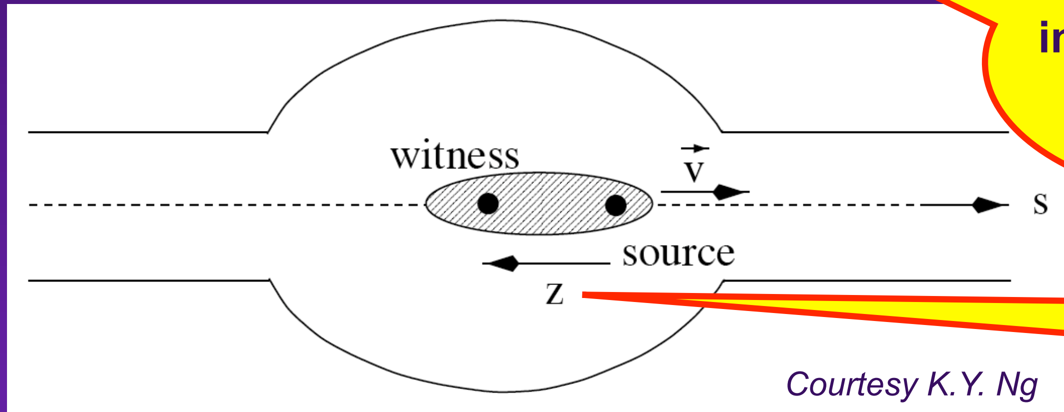
PANOFSKY-WENZEL THEOREM (1/2)

- **Proof of the Panofsky-Wenzel theorem**

$$\vec{\nabla}_\perp F_s = \frac{\partial \vec{F}_\perp}{\partial s}$$

Will be discussed in more detail in the "Wake fields and impedances"

$$\vec{\nabla}_\perp \equiv \begin{vmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{vmatrix}$$



Courtesy K.Y. Ng

Rigid-beam approximation

$$\overrightarrow{rot}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Rightarrow \quad \begin{vmatrix} \dfrac{\partial E_s}{\partial y} - \dfrac{\partial E_y}{\partial s} \\ \dfrac{\partial E_x}{\partial s} - \dfrac{\partial E_s}{\partial x} \\ \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} \end{vmatrix} = \begin{vmatrix} -\dfrac{\partial B_x}{\partial t} \\ -\dfrac{\partial B_y}{\partial t} \\ -\dfrac{\partial B_s}{\partial t} \end{vmatrix}$$

$$z = s - v\,t \quad \Rightarrow \quad \frac{d}{dz} = \frac{d}{ds} \qquad \frac{d}{dt} = -v\,\frac{d}{dz}$$

$$\Rightarrow \quad \frac{\partial E_s}{\partial x} = \frac{\partial}{\partial z}\left(E_x - v\,B_y\right) \quad \text{and} \quad \frac{\partial E_s}{\partial y} = \frac{\partial}{\partial z}\left(E_y + v\,B_x\right)$$

PANOFSKY-WENZEL THEOREM (2/2)

- **Computation of the electric field (in cylindrical coordinates)**

$$\vec{\nabla}_\perp F_s = \frac{\partial \vec{F}_\perp}{\partial s}$$

=>

$$\frac{\partial E_s}{\partial r} = \frac{\partial}{\partial z}\left(E_r - v\, B_\vartheta \right)$$

$$B_\vartheta = \frac{\beta}{c} E_r$$

=>

$$\boxed{\frac{\partial E_s}{\partial r} = \frac{1}{\gamma^2}\frac{\partial E_r}{\partial z}}$$

=>

$$\boxed{E_s\left(r=0\right) = -\frac{1}{\gamma^2}\frac{\partial}{\partial z}\int_0^b E_r\, dr}$$

$E_s\left(r=b\right)=0$ for a Perfectly Conducting (PC) beam pipe

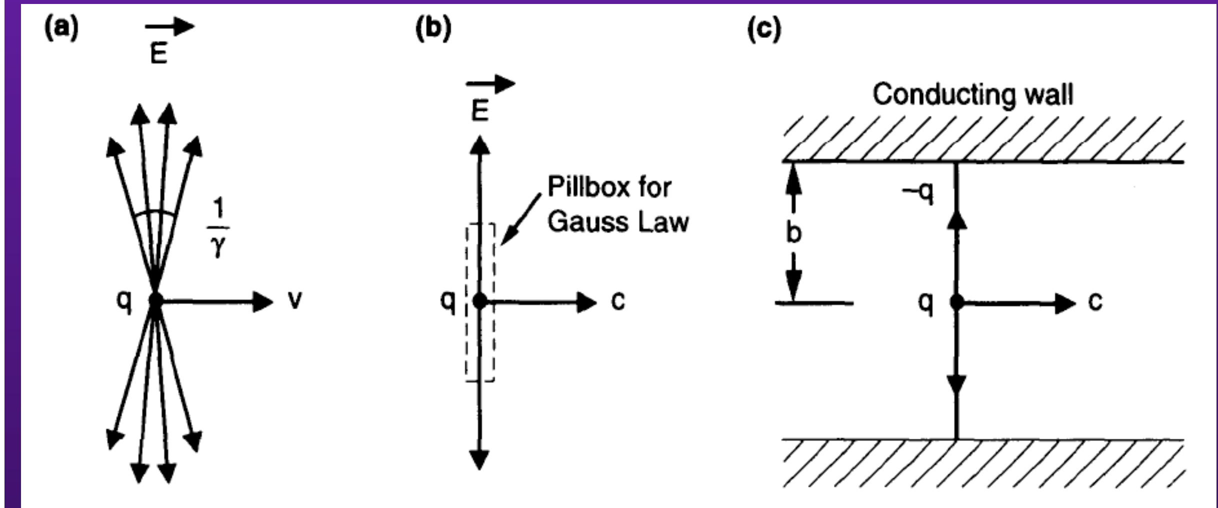
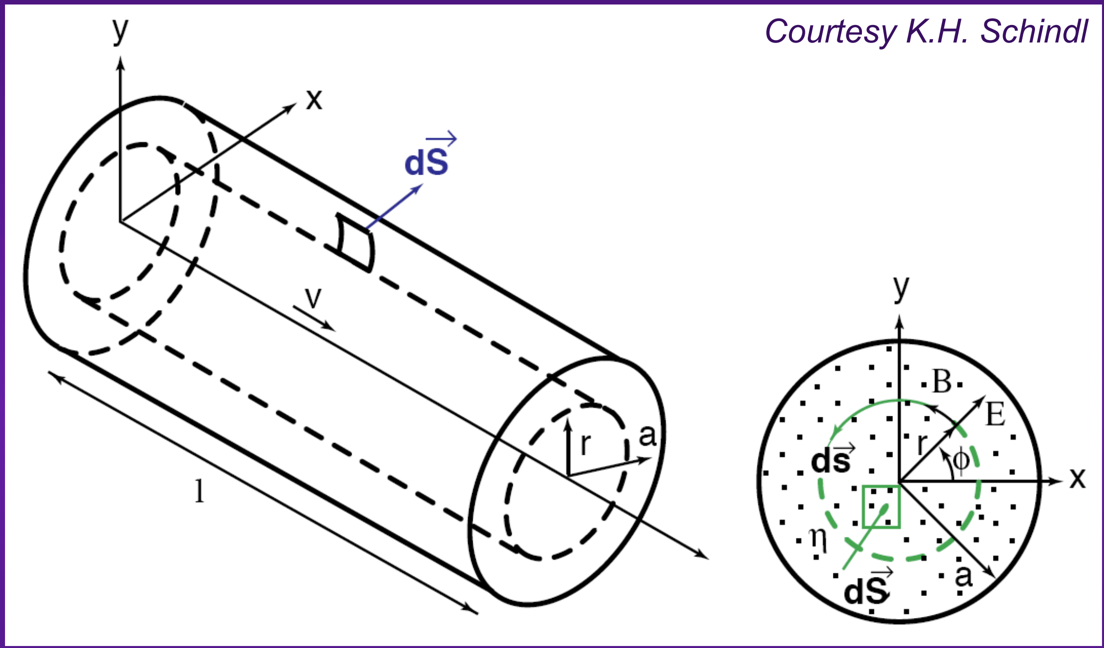


Figure 1.2. Electromagnetic field carried by an ultrarelativistic point charge: (a), (b) in free space; (c) in a perfectly conducting smooth pipe. *Courtesy A.W. Chao*

Due to symmetry (cylindrical beam pipe) => Only E_r, E_s and B_θ

EM FIELDS OF A CYLINDER WITH UNIFORM DENSITY (1/3)

- **EM fields of a cylinder with uniform density (with radius a) inside a beam pipe of radius b**



- **Charge density [C/m³]** $\rho = \dfrac{q}{\pi\, a^2\, l}$

$q = N_b\, e$

- **Current density [A/m²]** $J = \rho\, v$

- **Line density [C/m]** $\lambda_0 = \dfrac{q}{l}$

- **Total Current [A]** $I = \lambda_0\, v$

EM FIELDS OF A CYLINDER WITH UNIFORM DENSITY (2/3)

- $$\iiint div\, \vec{E}\, dV = \iint \vec{E} \, . \, d\vec{S} = \frac{1}{\varepsilon} \iiint \rho\, dV$$

=>

$$E_r\, 2\,\pi\, r\, l = \frac{1}{\varepsilon_0}\, \rho\, \pi\, r^2\, l \quad \text{for} \quad r < a$$

$$E_r\, 2\,\pi\, r\, l = \frac{1}{\varepsilon_0}\, \rho\, \pi\, a^2\, l \quad \text{for} \quad a < r < b$$

=>

$$\boxed{\begin{aligned} E_r &= \frac{\lambda(z)}{2\,\pi\,\varepsilon_0}\, \frac{r}{a^2} \quad \text{for} \quad r < a \\ E_r &= \frac{\lambda(z)}{2\,\pi\,\varepsilon_0}\, \frac{1}{r} \quad \text{for} \quad a < r < b \end{aligned}}$$

Generalization

$$\lambda_0 \rightarrow \lambda(z)$$

=> The (radial) Lorentz force on a particle of charge e inside the uniform cylinder is

$$F_r = \frac{e}{\gamma^2}\, E_r = \frac{e}{2\,\pi\,\varepsilon_0\,\gamma^2}\, \lambda(z)\, \frac{r}{a^2}$$

EM FIELDS OF A CYLINDER WITH UNIFORM DENSITY (3/3)

- **The (longitudinal) Lorentz force on a particle of charge e inside the uniform cylinder (on r = 0) is**

$$F_s\left(r=0\right) = -\frac{e}{2\pi\varepsilon_0\gamma^2}\frac{d\lambda(z)}{dz}\left(\int_0^a \frac{r}{a^2}dr + \int_a^b \frac{1}{r}dr\right)$$

=>

$$\boxed{F_s\left(r=0\right) = -\frac{e}{4\pi\varepsilon_0\gamma^2}\frac{d\lambda(z)}{dz}\left[1+2\ln\left(\frac{b}{a}\right)\right]}$$

EM FIELDS FOR A BUNCH WITH GAUSSIAN DENSITIES IN r and s (1/3)

- **EM fields and associated Lorentz force (for r < a) for a non-uniform bunch with Gaussian densities in r and s**

$$\rho\left(r,z\right)=\frac{1}{2\pi\sigma_r^2}\,e^{-\frac{r^2}{2\sigma_r^2}}\,\lambda\left(z\right)$$

$$\lambda\left(z\right)=\frac{q}{\sqrt{2\pi}\;\sigma_z}\,e^{-\frac{z^2}{2\sigma_z^2}}$$

- $$\iiint div\;\vec{E}\;dV=\iint\vec{E}\,.\,d\vec{S}=\frac{1}{\varepsilon}\iiint\rho\;dV$$

=> $$E_r\;2\,\pi\;r\;ds=\frac{\lambda\left(z\right)dz}{\varepsilon_0}\int_{\vartheta=0}^{2\pi}\int_{r'=0}^{r}\frac{e^{-\frac{r'^2}{2\sigma_r^2}}\;r'\,dr'}{2\pi\sigma_r^2}\,d\vartheta$$

=> $$\boxed{F_r=\frac{e}{\gamma^2}\,E_r=\frac{e\,\lambda\left(z\right)}{2\,\pi\;\varepsilon_0\;\gamma^2}\left(\frac{1-e^{-\frac{r^2}{2\sigma_r^2}}}{r}\right)}$$

Same result as for uniform case with $a=\sqrt{2}\;\sigma_r$

$$F_r\approx\frac{e\,\lambda\left(z\right)}{2\,\pi\;\varepsilon_0\;\gamma^2}\,\frac{r}{2\sigma_r^2}\qquad\text{for}\quad r<<\sigma_r$$

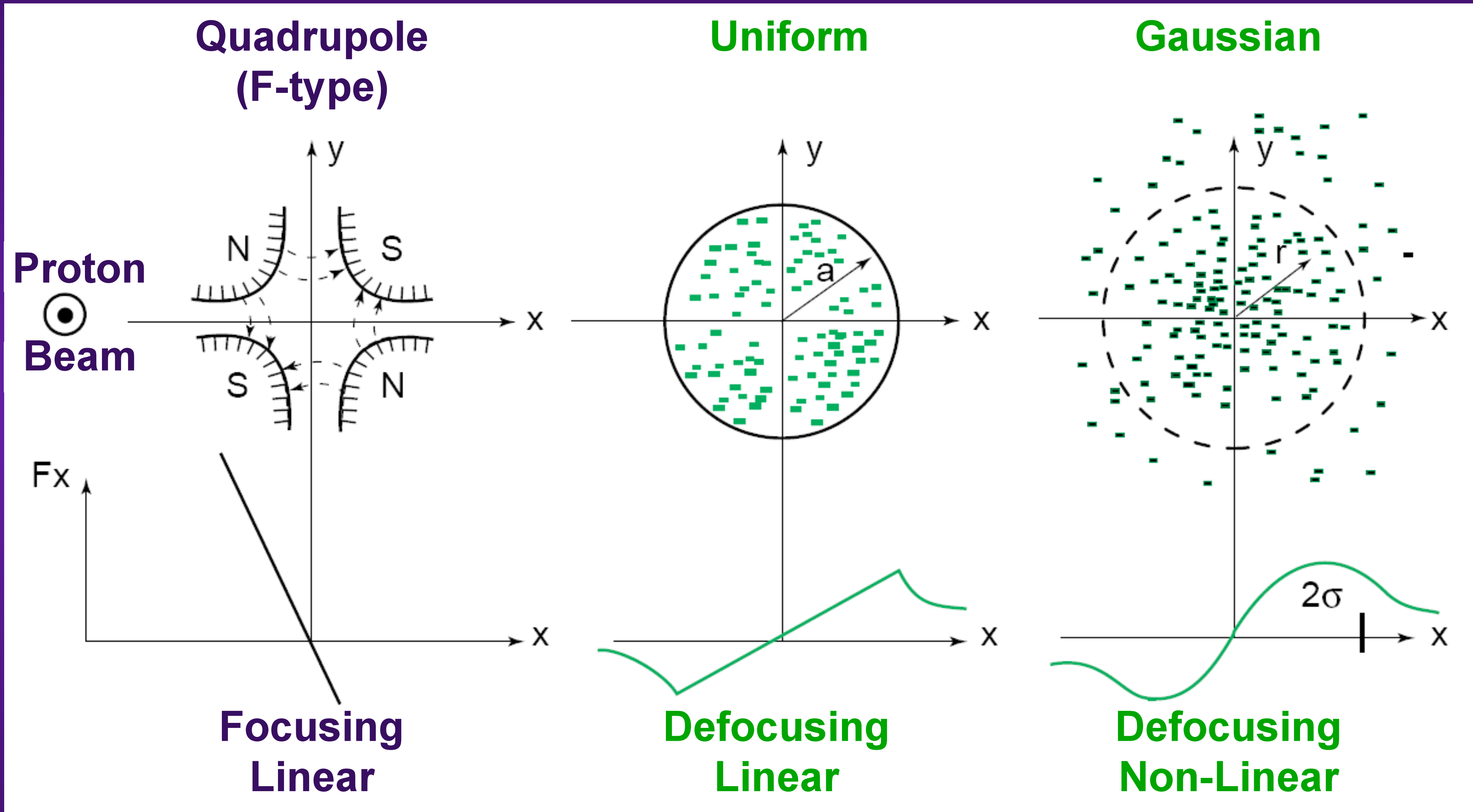
EM FIELDS FOR A BUNCH WITH GAUSSIAN DENSITIES IN r and s (2/3)

- **The associated (longitudinal) Lorentz force on a particle of charge e is**

$$\boxed{F_s(r) = -\frac{e}{2\pi\varepsilon_0\gamma^2}\frac{d\lambda(z)}{dz}\int_{r'=r}^{b}\frac{1-e^{-\frac{r'^2}{2\sigma_r^2}}}{r'}dr'}$$

Using $\dfrac{\partial E_s}{\partial r} = \dfrac{1}{\gamma^2}\dfrac{\partial E_r}{\partial z}$

EM FIELDS FOR A BUNCH WITH GAUSSIAN DENSITIES IN r and s (3/3)



Courtesy K. Schindl

- **Transverse incoherent tune shift induced by the "direct" SC**

  - **Equation of motion** $$\frac{d^2 x}{ds^2} + K_x(s)\, x = \frac{F_x}{\beta^2 E_{total}}$$ F_x^{pert}

  - **Linearizing (for a transversally Gaussian bunch)** $$F_x = \frac{e\,\lambda(z)}{2\,\pi\,\varepsilon_0\,\gamma^2}\,\frac{x}{2\sigma_x^2} \quad \text{for} \quad x << \sigma_x$$

=> $$\frac{d^2 x}{ds^2} + \left[K_x(s) + K_{SC,x}(z)\right] x = 0$$

with $$K_{SC,x}(z) = -\frac{e\,\lambda(z)}{4\,\pi\,\varepsilon_0\,E_{total}\,\beta^2\,\gamma^2\,\sigma_x^2}$$

$$\Delta Q_x \equiv \frac{1}{4\pi}\int_{s=0}^{2\pi R} K_{SC,x}(z)\,\beta_x(s)\,ds$$ **=>** $$\Delta Q_x = -\frac{e\,R\,\lambda(z)}{8\,\pi\,\varepsilon_0\,E_{total}\,\beta\,\gamma\,\varepsilon_{x,rms}^{norm}}$$

with $\varepsilon^{norm}_{x,rms} = \beta \gamma \varepsilon_{x,rms}$ $\varepsilon_{x,rms} = \frac{\sigma_x^2}{\beta_x}$

At a place of 0 dispersion

- Assuming then $\lambda(z) = \frac{N_b \, e}{\sqrt{2\pi} \, \sigma_z} \, e^{-\frac{z^2}{2\sigma_z^2}}$

=> $$\Delta Q_x^{\max} = - \frac{N_b \, R \, r_p}{2 \sqrt{2\pi} \, \beta \gamma^2 \sigma_z \, \varepsilon^{norm}_{x,rms}}$$

=> $$\boxed{\Delta Q_x^{\max} = - \frac{N_b \, r_p}{4 \pi \beta \gamma^2 \varepsilon^{norm}_{x,rms} \, B}}$$

where *B* is the bunching factor defined by

$$B = \frac{I_{average}}{I_{peak}} = \frac{\sqrt{2\pi} \, \sigma_z}{2 \pi R}$$

SC is always defocusing in the transverse planes

B = 1 for a coasting beam

TRANSVERSE AND LONGITUDINAL INCOHERENT TUNE SHIFTS (3/10)

- **General definition of the (local) bunching factor**

Total number of particles $= N = N_b \, M$ for M bunches

$$\boxed{\frac{1}{B(z)} = 2\pi R \, \frac{1}{N} \, \frac{dN}{dz} = \frac{I_{local}}{I_{average}}}$$

- **For a Gaussian line density**

$$\frac{1}{N}\frac{dN}{dz} = \frac{1}{\sqrt{2\pi}\,\sigma_z \, M} \, e^{-\frac{z^2}{2\sigma_z^2}}$$

$$\Rightarrow \quad \frac{1}{B} = \frac{1}{B(0)} = \frac{2\pi R}{\sqrt{2\pi}\,\sigma_z \, M}$$

- **For a parabolic line density**

$$\frac{1}{N}\frac{dN}{dz} = \frac{3}{2 L M}\left[1 - \left(\frac{2z}{L}\right)^2\right]$$

$$\Rightarrow \quad \frac{1}{B} = \frac{1}{B(0)} = \frac{3\pi R}{L M}$$

TRANSVERSE AND LONGITUDINAL INCOHERENT TUNE SHIFTS (4/10)

- **Another way to deduce the transverse incoherent tune shift induced by the "direct" SC**
  - **Equation of motion** $$\frac{d^2 x}{ds^2} + K_x(s)\, x = \frac{F_x}{\beta^2 E_{total}}$$
  - **Smooth approximation** $$K_x(s) = \left(\frac{Q_{x0}}{R}\right)^2$$

=> $$\frac{d^2 x}{ds^2} + \frac{1}{R^2}\left(Q_{x0}^2 - \frac{e R^2 \lambda(z)}{4\,\pi\,\varepsilon_0\, E_{total}\,\beta^2\gamma^2\sigma_x^2}\right) x = 0$$

$$\left(Q_{x0} + \Delta Q_x\right)^2 \approx Q_{x0}^2 + 2\, Q_{x0}\, \Delta Q_x$$

=> $$\Delta Q_x = -\frac{e R^2 \lambda(z)}{4\,\pi\,\varepsilon_0\, E_{total}\,\beta^2\gamma^2\sigma_x^2} \times \frac{1}{2\, Q_{x0}}$$

New tune: $Q_x = Q_{x0} + \Delta Q_x$

It is the same result as before

TRANSVERSE AND LONGITUDINAL INCOHERENT TUNE SHIFTS (5/10)

- **Case of an elliptical beam**
  - **Particle density** $n(x,y) = n\left(\frac{x^2}{x_m^2} + \frac{y^2}{y_m^2}\right)$ **, i.e. elliptical symmetry, and 0 outside the ellipse** $\frac{x^2}{x_m^2} + \frac{y^2}{y_m^2} = 1$

=> It can be shown that

$$\boxed{E_x = \frac{e\, x_m\, y_m\, x}{2\varepsilon_0} \int_{s=0}^{s=+\infty} n\left(\frac{x^2}{x_m^2 + s} + \frac{y^2}{y_m^2 + s}\right)\left(x_m^2 + s\right)^{-3/2}\left(y_m^2 + s\right)^{-1/2} ds}$$

$$\boxed{E_y = \frac{e\, x_m\, y_m\, y}{2\varepsilon_0} \int_{s=0}^{s=+\infty} n\left(\frac{x^2}{x_m^2 + s} + \frac{y^2}{y_m^2 + s}\right)\left(x_m^2 + s\right)^{-1/2}\left(y_m^2 + s\right)^{-3/2} ds}$$

- **Let's assume first a constant density** $n(x,y) = \dfrac{N_1}{\pi\, a\, b}$

N_1 is the number of particles / unit length (= N / 2 π R for a continuous beam)

=> $$E_x = \frac{e N_1}{\pi \varepsilon_0} \frac{x}{x_m \left(x_m + y_m \right)} \qquad E_y = \frac{e N_1}{\pi \varepsilon_0} \frac{y}{y_m \left(x_m + y_m \right)}$$

- **Reminder: For the case of a circular beam ($x_m = y_m = a$) with constant density we found (e.g. in y-plane)**

$$E_y^{Const} = \frac{e N_1}{\pi \varepsilon_0} \frac{y}{2 y_m^2}$$

=> **To go from a round to an elliptical beam, replace $2 y_m^2$ by $y_m \left(x_m + y_m \right)$ (and $2 x_m^2$ by $x_m \left(x_m + y_m \right)$)**

TRANSVERSE AND LONGITUDINAL INCOHERENT TUNE SHIFTS (7/10)

- **Let's assume now a parabolic density**

$$n(x,y) = \frac{2\,N_1}{\pi\,a\,b}\left(1 - \frac{x^2}{x_m^2} - \frac{y^2}{y_m^2}\right)$$

***N*$_1$ is the number of particles / unit length**
(= *N* / 2 π R for a continuous beam)

The integrals can be evaluated by changing the variable, using *u* given by (see "Introduction")

$$u^2 = b^2 + s$$

=>

$$E_x = \frac{2\,e\,N_1}{\pi\,\varepsilon_0}\left[x\,\frac{1}{x_m\left(x_m + y_m\right)} - x^3\,\frac{2\,x_m + y_m}{3\,x_m^3\left(x_m + y_m\right)^2} - x\,y^2\,\frac{1}{x_m\,y_m\left(x_m + y_m\right)^2}\right]$$

$$E_y = \frac{2\,e\,N_1}{\pi\,\varepsilon_0}\left[y\,\frac{1}{y_m\left(x_m + y_m\right)} - y^3\,\frac{2\,y_m + x_m}{3\,y_m^3\left(x_m + y_m\right)^2} - y\,x^2\,\frac{1}{x_m\,y_m\left(x_m + y_m\right)^2}\right]$$

TRANSVERSE AND LONGITUDINAL INCOHERENT TUNE SHIFTS (8/10)

- **Linearizing, we obtain (for instance in the y-plane)**

$$E_y \approx \frac{e\,N_1}{\pi\,\varepsilon_0}\,\frac{2\,y}{y_m\left(x_m + y_m\right)}$$

✧ **Reminder: For the case of a bi-Gaussian beam, we had**

$$E_y^{G,lin} \approx \frac{e\,N_1}{\pi\,\varepsilon_0}\,\frac{y}{\left(2\sigma_y\right)^2}$$

=> The same result is obtained for the case of a round beam ($x_m = y_m$) if $y_m = 2\,\sigma_y$

- **Longitudinal tune shift from SC**

  - **Equation of motion** $$\frac{d^2 z}{ds^2} + \left(\frac{Q_s}{R}\right)^2 z = -\eta \frac{F_s}{\beta^2 E_{total}}$$

  - **Linearizing (for a transversally Gaussian bunch)** $$F_s = -\frac{e}{2\pi\varepsilon_0\gamma^2}\frac{d\lambda(z)}{dz}\left(\int_r^{a=\sqrt{2}\,\sigma_r}\frac{r'}{2\sigma_r^2}dr' + \int_a^b \frac{dr'}{r'}\right)$$

=> $$F_s = -\frac{e}{4\pi\varepsilon_0\gamma^2}\frac{d\lambda(z)}{dz}\left[1 + 2\ln\left(\frac{b}{a}\right)\right]$$

=> $$\frac{d^2 z}{ds^2} + \left(\frac{Q_{s0}}{R}\right)^2 z = \frac{\eta\, e}{4\pi\varepsilon_0 E_{total}\beta^2\gamma^2}\frac{d\lambda(z)}{dz}\left[1 + 2\ln\left(\frac{b}{a}\right)\right]$$

As it is the same result as for uniform case with $a = \sqrt{2}\,\sigma_r$

TRANSVERSE AND LONGITUDINAL INCOHERENT TUNE SHIFTS (10/10)

- **Assuming then** $$\lambda(z) = \frac{N_b\, e}{\sqrt{2\pi}\ \sigma_z}\, e^{-\frac{z^2}{2\sigma_z^2}}$$

=> $$\frac{d\lambda(z)}{d\,z} = -\frac{z}{\sigma_z^2}\,\lambda(z) \approx -z\,\frac{N_b\, e}{\sqrt{2\pi}\ \sigma_z^3} \quad \text{for} \quad z \ll \sigma_z$$

=> $$\frac{d^2 z}{ds^2} + \frac{1}{R^2}\left(Q_{s0}^2 + \frac{\eta\, N_b\, e^2\, R^2}{4\pi\sqrt{2\pi}\ \varepsilon_0\, E_{total}\, \beta^2\, \gamma^2\, \sigma_z^3}\left[1 + 2\ln\left(\frac{b}{a}\right)\right]\right) z = 0$$

$$\left(Q_{s0} + \Delta Q_s\right)^2 \approx Q_{s0}^2 + 2\, Q_{s0}\, \Delta Q_s$$

=> $$\boxed{\Delta Q_s = \frac{\eta\, N_b\, e^2\, R^2}{8\pi\sqrt{2\pi}\ \varepsilon_0\, E_{total}\, \beta^2\, \gamma^2\, \sigma_z^3\, Q_{s0}}\left[1 + 2\ln\left(\frac{b}{a}\right)\right]}$$

New tune: $Q_s = Q_{s0} + \Delta Q_s$

In the longitudinal plane, it is found that SC is defocusing Below Transition (BT) and focusing above (AT)

TRANSVERSE TUNE SPREADS (1/6)

- **Transverse tune spread (due to the nonlinear force)**

  - **Let's assume the following particle density (considering a round beam, $\sigma_x = \sigma_y = \sigma$)**

$$x_m = \sqrt{10}\;\sigma \approx 3.2\,\sigma$$

$$n(x,y) = n_0 \left(1 - \frac{x^2 + y^2}{x_m^2} \right)^3$$

with

$$n_0 = \frac{2\,N_b}{B\,\pi^2 R\, x_m^2}$$

Assuming a Gaussian longitudinal profile

$$B = \sqrt{2\pi}\;\sigma_z / (2\pi R)$$

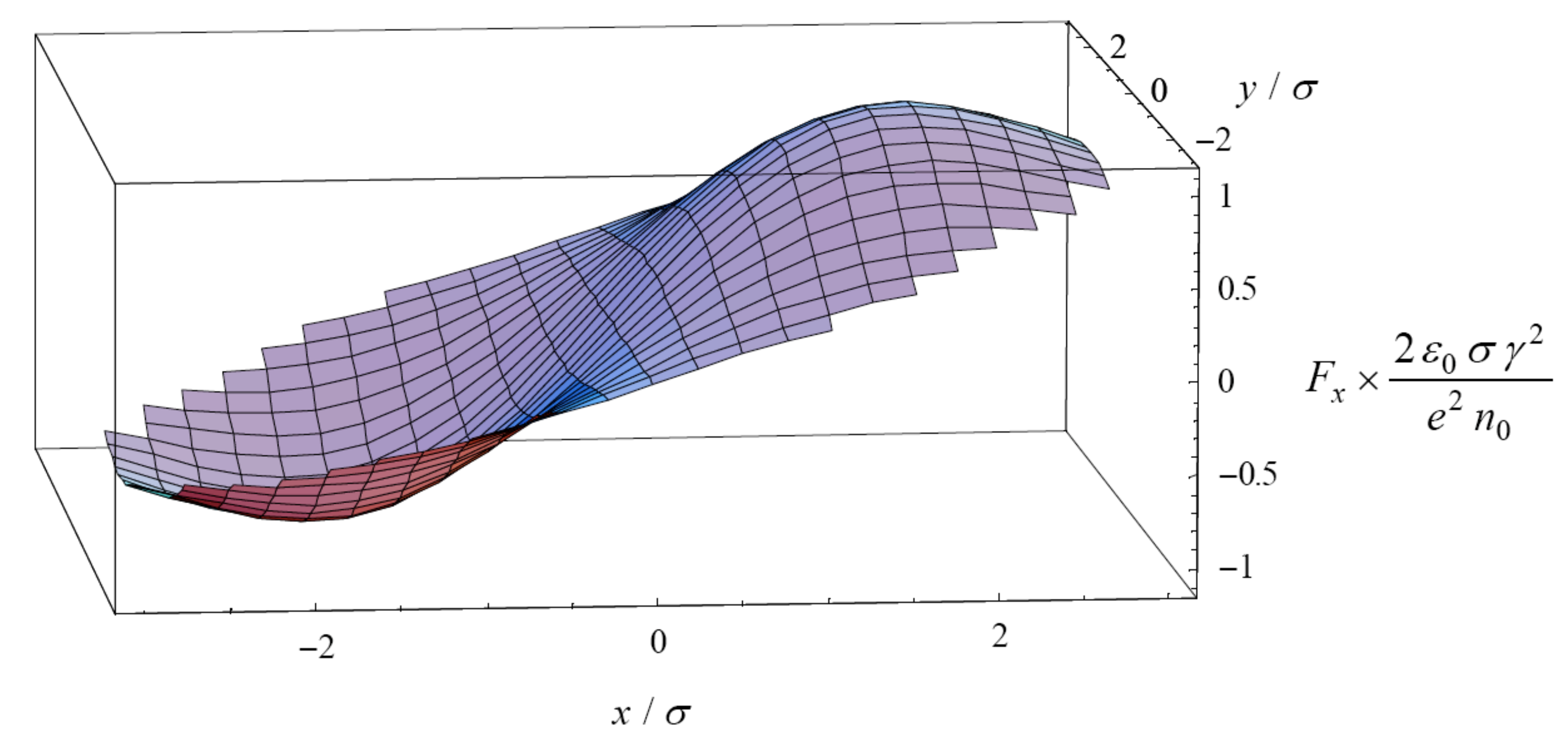
$$\int_x \int_y n(x,y) = \frac{N}{2\pi R}$$

For a coasting beam

=>

$$\boxed{F_x = \frac{e\,E_x}{\gamma^2} = \frac{e^2 n_0}{2\varepsilon_0 \gamma^2} \left[x - \frac{3x\left(x^2+y^2\right)}{2\,x_m^2} + \frac{x\left(x^2+y^2\right)^2}{x_m^4} - \frac{x\left(x^2+y^2\right)^3}{4\,x_m^6} \right]}$$

TRANSVERSE TUNE SPREADS (2/6)



TRANSVERSE TUNE SPREADS (3/6)

- **(Non-linear) space-charge tune shift:** For an approximate solution, the non-linear dependence of the force is converted into an amplitude dependence of the particle's tune using the method of the harmonic balance, which is an averaging process over the incoherent betatron motions

Action variables

$x = x_0 \cos\varphi$ $\quad y = y_0 \cos\vartheta$ $\quad x_0 = \sqrt{2 J_x}$ $\quad y_0 = \sqrt{2 J_y}$

=>

$$< x^3 > \approx \frac{3}{4} x_0^2 \, x \,, \qquad < x \, y^2 > \approx \frac{1}{2} y_0^2 \, x \,, \qquad < x^5 > \approx \frac{5}{8} x_0^4 \, x \,,$$

$$< x^3 y^2 > \approx \frac{3}{8} x_0^2 \, y_0^2 \, x \,, \qquad < x \, y^4 > \approx \frac{3}{8} y_0^4 \, x \,, \qquad < x^7 > \approx \frac{35}{64} x_0^6 \, x \,,$$

$$< x^5 y^2 > \approx \frac{5}{16} x_0^4 \, y_0^2 \, x \,, \qquad < x^3 y^4 > \approx \frac{9}{32} x_0^2 \, y_0^4 \, x \,, \qquad < x \, y^6 > \approx \frac{5}{16} y_0^6 \, x \,,$$

TRANSVERSE TUNE SPREADS (4/6)

=> $$\Delta Q^{x}_{incoh}\left(j_x, j_y\right) = \Delta_0 \left[1 - \frac{9}{8} j_x - \frac{3}{4} j_y + \frac{5}{8} j_x^2 + \frac{3}{4} j_x\, j_y + \frac{3}{8} j_y^2 - \frac{35}{256} j_x^3 - \frac{15}{64} j_x^2\, j_y - \frac{27}{128} j_x\, j_y^2 - \frac{5}{64} j_y^3 \right]$$

$$j_x = J_x / J_{\max} \qquad j_y = J_y / J_{\max}$$

with $$\Delta_0 = -\frac{N_b\, r_p}{5\, \pi\, B\, \beta\, \gamma^2 \varepsilon^{norm}_{rms}}$$

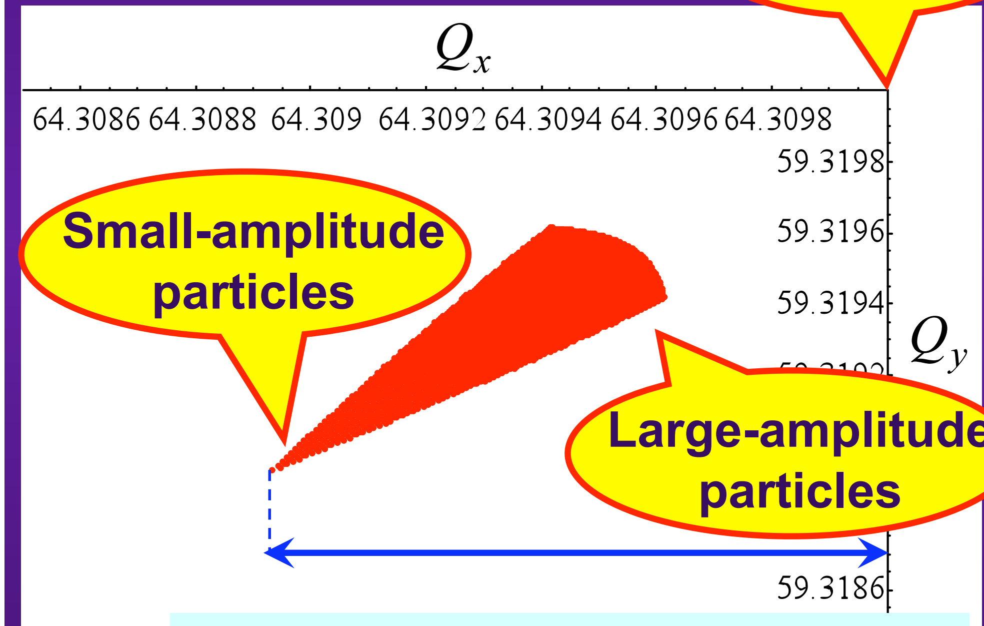
$$J_{\max} = 5\, \sigma^2 \qquad \varepsilon^{norm}_{rms} = \beta\, \gamma\, \varepsilon$$

It was 4 in the case of a bi-Gaussian

TRANSVERSE TUNE SPREADS (5/6)

- **2D TUNE FOOTPRINT**

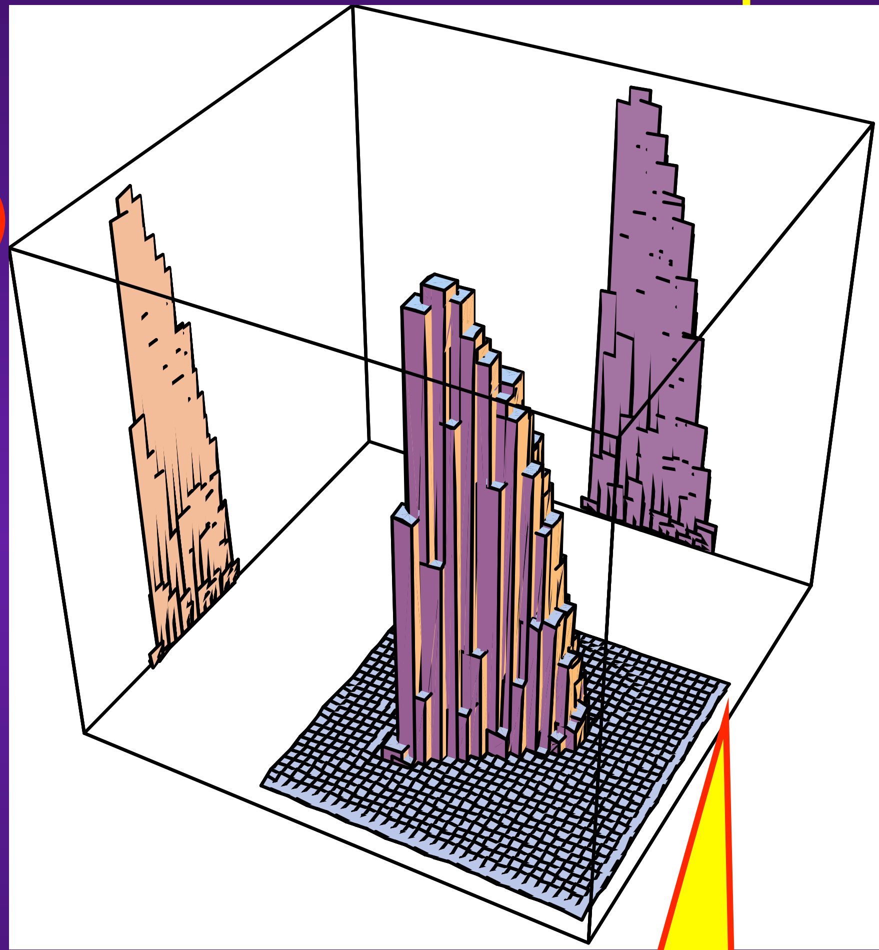
  ⇒ **INCOHERENT (single-particle) tunes**



$$\Delta_0 \propto -\frac{N_b}{\beta \gamma^2 \varepsilon_{rms}^{norm}}$$

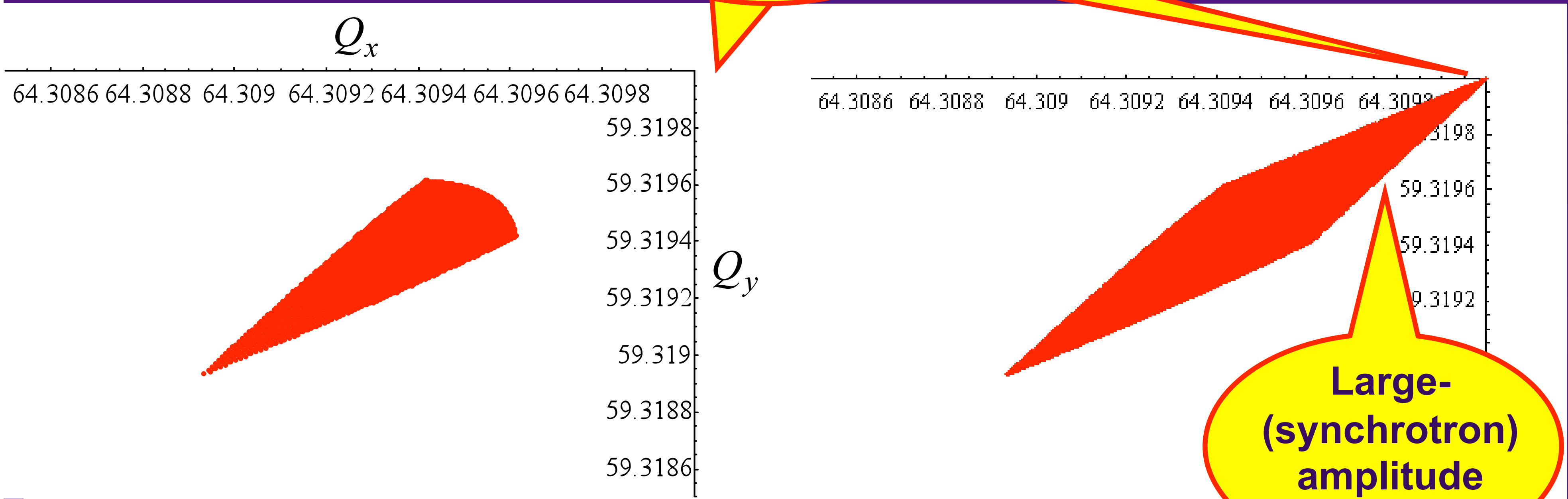
= Linear space - charge tune shift

- **3D view of the 2D tune footprint**



TRANSVERSE TUNE SPREADS (6/6)

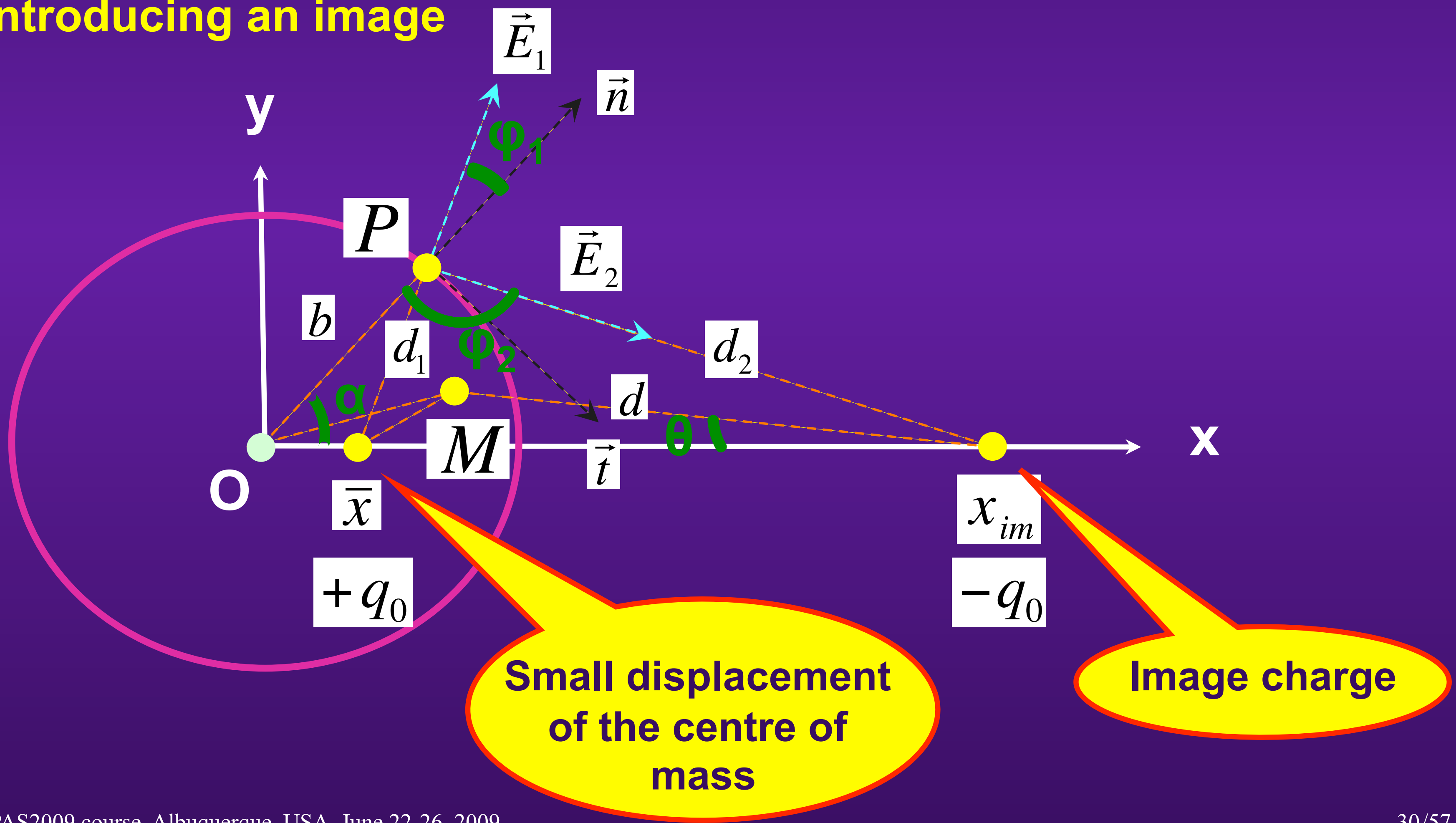
- **2D tune footprint**
- **3D tune footprint**



=> The longitudinal variation (due to synchrotron oscillations) of the transverse space-charge force fills the gap until the low-intensity working point

Considering B (s) and not only B (0)

- **Effect of the images (i.e. the wall) in the case of a beam off-axis in a (Perfectly Conducting, PC) circular beam pipe**
  - **The boundary condition on a PC ($E_t = 0$) is satisfied by introducing an image**



BEAM OFF-AXIS IN A PC CIRCULAR BEAM PIPE (2/5)

- $$E_1 = \frac{\lambda}{2\,\pi\,\varepsilon_0\,d_1}$$
  $$E_2 = \frac{\lambda}{2\,\pi\,\varepsilon_0\,d_2}$$

- $$E_{1t} = -\,E_1 \sin\varphi_1$$
  $$E_{2t} = E_2 \sin(\pi - \varphi_2) = E_2 \sin\varphi_2$$

- $$E_t = 0 \quad \Rightarrow \quad E_{1t} + E_{2t} = 0$$ **=>** $$\frac{\sin\varphi_1}{d_1} = \frac{\sin\varphi_2}{d_2}$$

- **General relations in a triangle (see “Introduction”)**

$$\frac{x_{im}}{\sin\varphi_2} = \frac{d_2}{\sin\alpha}$$

$$d_1^2 = \bar{x}^2 + b^2 - 2\,\bar{x}\,b\cos\alpha$$

$$\frac{\bar{x}}{\sin\varphi_1} = \frac{d_1}{\sin\alpha}$$

$$d_2^2 = x_{im}^2 + b^2 - 2\,x_{im}\,b\cos\alpha$$

BEAM OFF-AXIS IN A PC CIRCULAR BEAM PIPE (3/5)

=> $$\frac{\bar{x}}{d_1^2} = \frac{x_{im}}{d_2^2}$$ and $$\left(\frac{x_{im}}{\bar{x}}\right)^2 - \left[1 + \left(\frac{b}{\bar{x}}\right)^2\right] \frac{x_{im}}{\bar{x}} + \left(\frac{b}{\bar{x}}\right)^2 = 0$$

=> $x_{im} = \bar{x}$ or $x_{im} = \dfrac{b^2}{\bar{x}}$

=> The correct position for the image charge ($-q_0$) is $$\boxed{x_{im} = \frac{b^2}{\bar{x}}}$$

- To compute the image electric force, we place a witness line charge (λ) at point M (x,y). The electric force is

$$\frac{F_x^{ele}}{e} = E_x = \frac{\lambda}{2\,\pi\,\varepsilon_0\,d}\cos\vartheta$$

BEAM OFF-AXIS IN A PC CIRCULAR BEAM PIPE (4/5)

- $$\cos\vartheta = \frac{x_{im} - x}{d}$$ **=>** $$\frac{\cos\vartheta}{d} = \frac{x_{im} - x}{d^2} = \frac{x_{im} - x}{\left(x_{im} - x\right)^2 + y^2}$$

With $$x_{im} = \frac{b^2}{\overline{x}}$$ **=>** $$\frac{F_x^{ele}}{e} = \frac{\lambda}{2\,\pi\,\varepsilon_0}\,\frac{\frac{b^2}{\overline{x}} - x}{\left(\frac{b^2}{\overline{x}} - x\right)^2 + y^2}$$

=> $$\boxed{\frac{F_x^{ele}}{e} \approx \frac{\lambda}{2\,\pi\,\varepsilon_0}\,\frac{\overline{x}}{b^2} \quad \text{for } \overline{x} \ll \text{b}}$$

BEAM OFF-AXIS IN A PC CIRCULAR BEAM PIPE (5/5)

- For the magnetic contribution, the situation is complicated by the fact that the magnetic field may or may not penetrate the vacuum chamber
- It is always assumed that the electric field is non-penetrating (as done before)
- For the magnetic field
  - High-frequency components will not penetrate => ac
  - Low-frequency components will penetrate and form images on the magnet pole faces (if there are some; otherwise they will go to infinity and will not act back on the beam) => dc

=> In the case of a non-penetrating magnetic field, one finally obtains

$$\boxed{F_x \approx \underbrace{\frac{\lambda\, e}{2\,\pi\,\varepsilon_0}}_{\text{Electric part}} \left(1 - \underbrace{\beta^2}_{\text{ac magnetic part}}\right) \frac{\bar{x}}{b^2} = \frac{\lambda\, e}{2\,\pi\,\varepsilon_0\,\gamma^2}\,\frac{\bar{x}}{b^2} \quad \text{for } \bar{x} << \mathrm{b}}$$

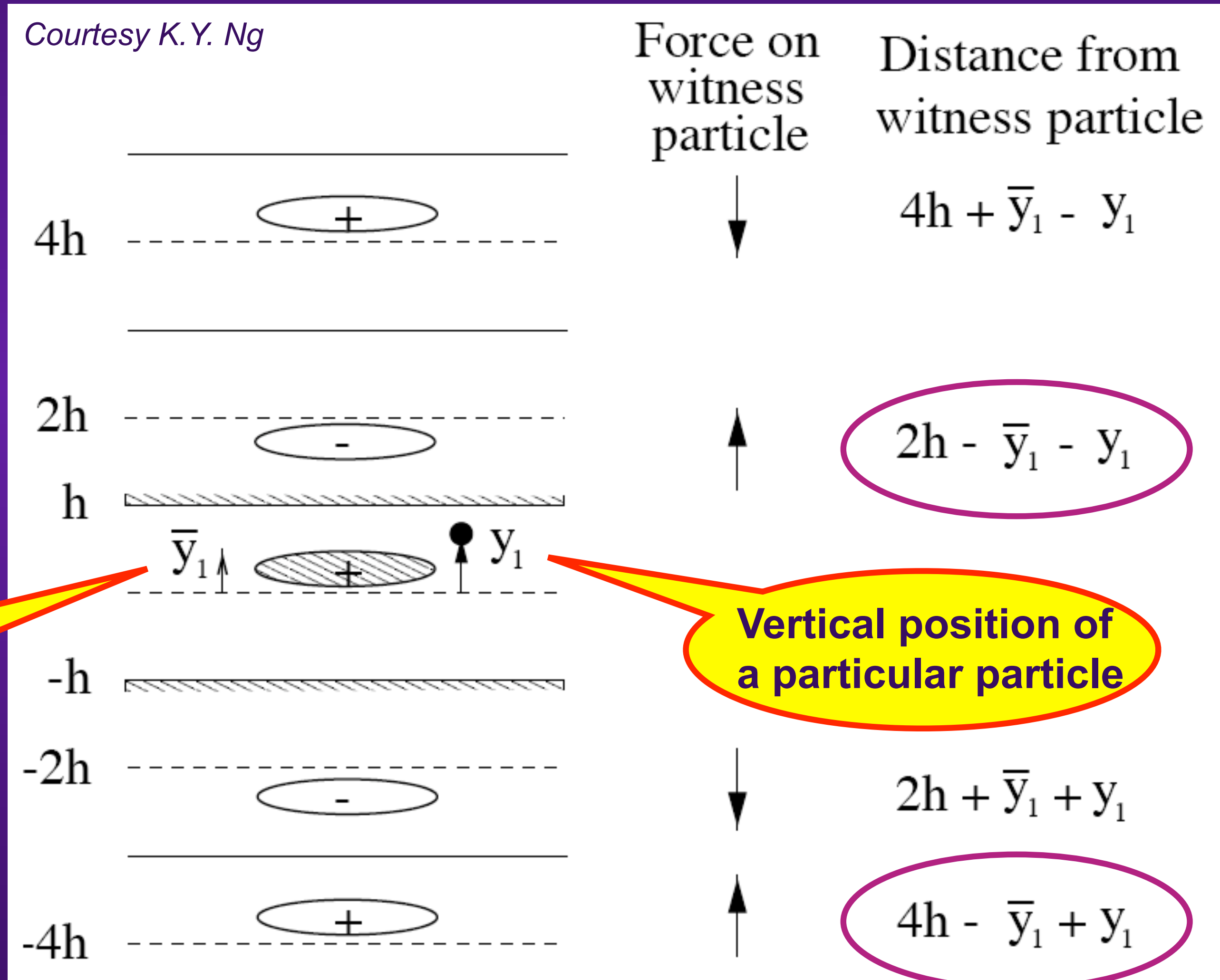
BEAM OFF-AXIS BETWEEN 2 INFINITE PC // PLATES (1/12)

- **Effect of the images (i.e. the wall) in the case of a beam off-axis between 2 infinite (Perfectly Conducting, PC) // plates**

- **ELECTRIC IMAGES**

=> Boundary condition on the plates:

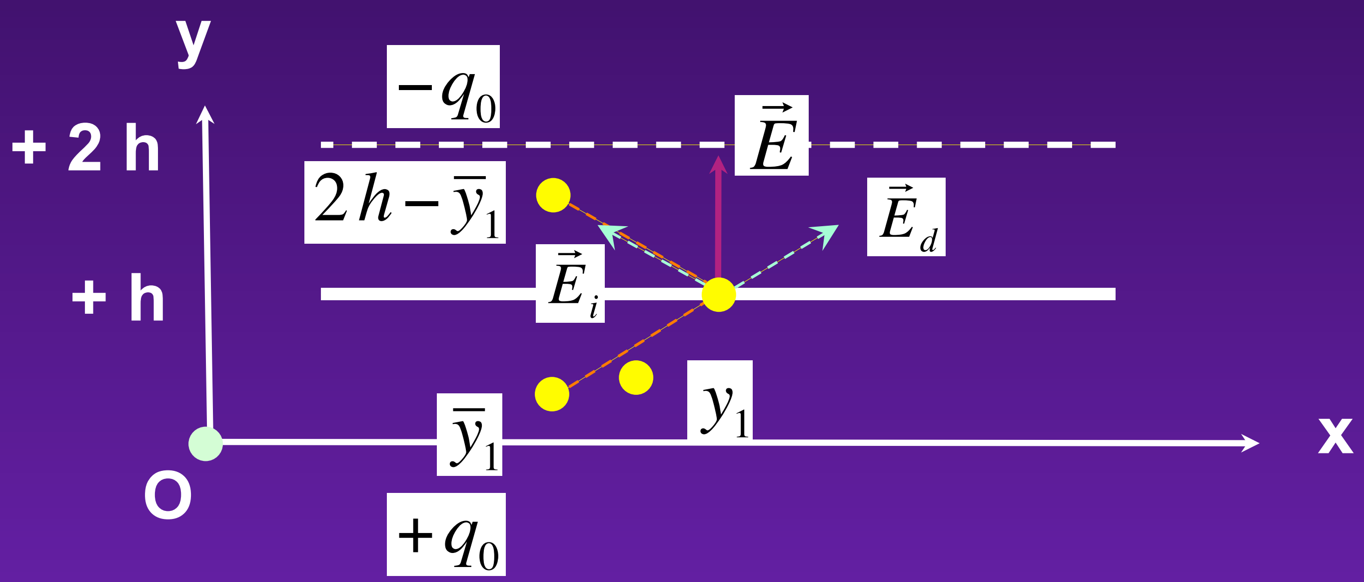
$$E_t = 0$$



Vertical offset of the whole beam

Vertical position of a particular particle

BEAM OFF-AXIS BETWEEN 2 INFINITE PC // PLATES (2/12)



=> $E_x = 0$ on the upper plate, as it should be for a PC wall

=> This (negative) image will have another (positive) image for the lower plate (and so on)

=> Finally, the same thing applies for the interaction between the beam and the lower plate

BEAM OFF-AXIS BETWEEN 2 INFINITE PC // PLATES (3/12)

=>

$$E_y = \frac{\lambda}{2\pi\varepsilon_0}\left[\begin{array}{l} + \dfrac{1}{2h - \bar{y}_1 - y_1} - \dfrac{1}{2h + \bar{y}_1 + y_1} + \dfrac{1}{6h - \bar{y}_1 - y_1} - \dfrac{1}{6h + \bar{y}_1 + y_1} + \ldots \\ - \dfrac{1}{4h + \bar{y}_1 - y_1} + \dfrac{1}{4h - \bar{y}_1 + y_1} - \dfrac{1}{8h + \bar{y}_1 - y_1} + \dfrac{1}{8h - \bar{y}_1 + y_1} + \ldots \end{array}\right]$$

Assuming h >> transverse beam sizes

=>

$$E_y = \frac{\lambda}{2\pi\varepsilon_0}\left[\begin{array}{l} + \dfrac{2(\bar{y}_1 + y_1)}{(2h)^2 - (\bar{y}_1 + y_1)^2} + \dfrac{2(\bar{y}_1 + y_1)}{(6h)^2 - (\bar{y}_1 + y_1)^2} + \ldots \\ + \dfrac{2(\bar{y}_1 - y_1)}{(4h)^2 - (\bar{y}_1 - y_1)^2} + \dfrac{2(\bar{y}_1 - y_1)}{(8h)^2 - (\bar{y}_1 - y_1)^2} + \ldots \end{array}\right]$$

=> Keeping only the linear terms in $\bar{y}_1$ **and** y_1 **, one has**

BEAM OFF-AXIS BETWEEN 2 INFINITE PC // PLATES (4/12)

$$E_y = \frac{\lambda}{2\,\pi\,\varepsilon_0}\left[\begin{array}{l} 2\left(\overline{y}_1 + y_1\right)\left\{\dfrac{1}{\left(2h\right)^2} + \dfrac{1}{\left(6h\right)^2} + \ldots\right\} \\ + 2\left(\overline{y}_1 - y_1\right)\left\{\dfrac{1}{\left(4h\right)^2} + \dfrac{1}{\left(8h\right)^2} + \ldots\right\} \end{array}\right]$$

=> $$E_y = \frac{\lambda}{\pi\,\varepsilon_0\,h^2}\left[\begin{array}{l} \left(\overline{y}_1 + y_1\right)\left\{\dfrac{1}{2^2} + \dfrac{1}{6^2} + \ldots\right\} \\ + \left(\overline{y}_1 - y_1\right)\left\{\dfrac{1}{4^2} + \dfrac{1}{8^2} + \ldots\right\} \end{array}\right]$$

=> $$\boxed{E_y = \frac{\lambda}{2\pi\,\varepsilon_0\,h^2}\left(\frac{\pi^2}{12}\,\overline{y}_1 + \frac{\pi^2}{24}\,y_1\right)}$$

One can define the Laslett coherent electric image coefficient ξ_{1y} (obtained when $\overline{y}_1 = y_1$), and $2\xi_{1y} = \pi^2 / 8$

It is called $2\varepsilon_{1y}$ in the literature = Laslett incoherent electric image coefficient

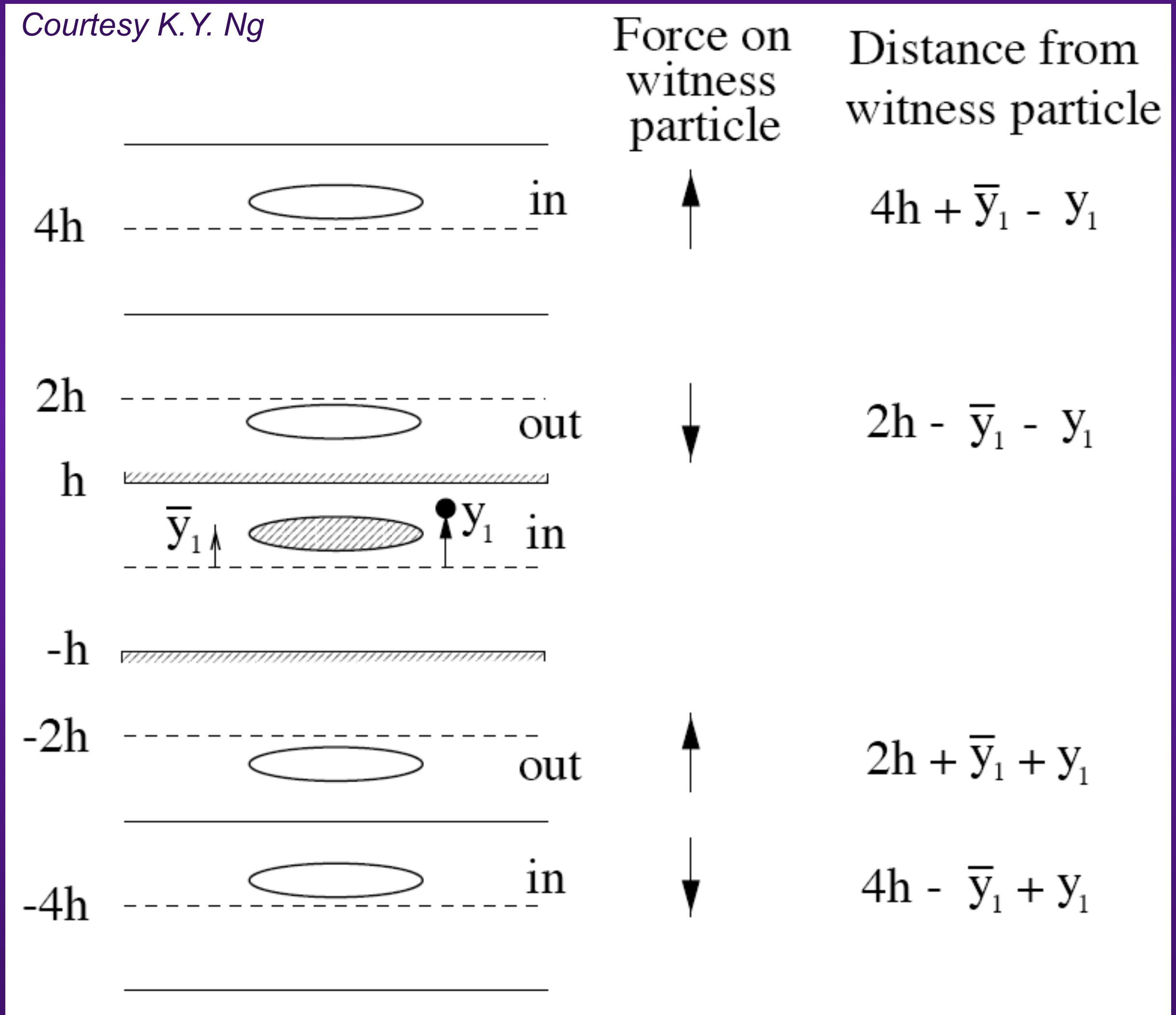
BEAM OFF-AXIS BETWEEN 2 INFINITE PC // PLATES (5/12)

- **MAGNETIC IMAGES (1)**

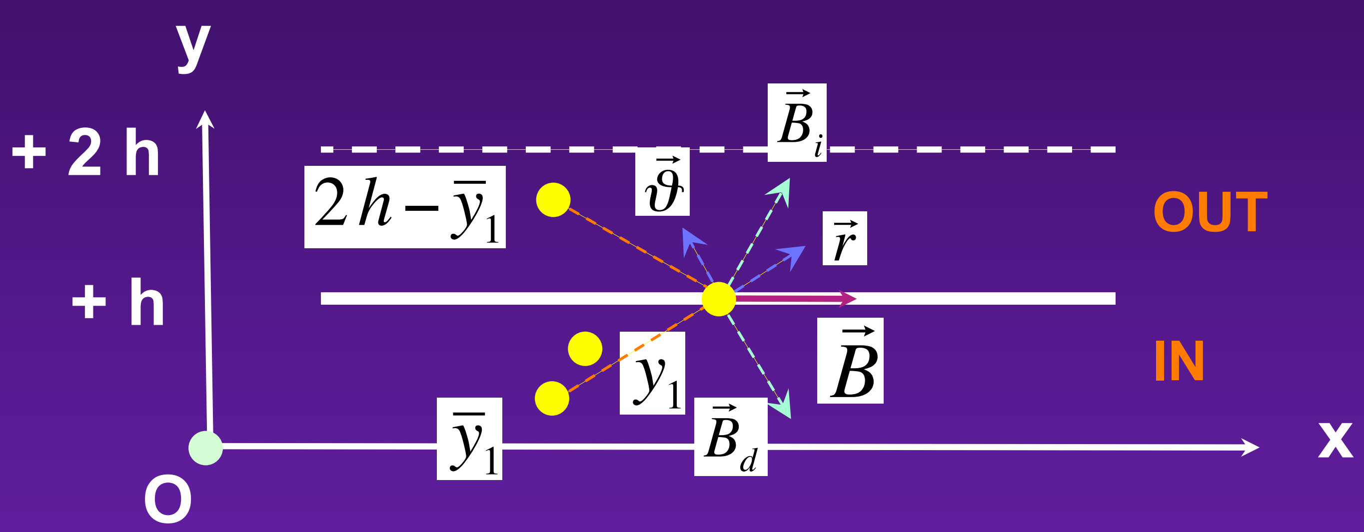
  (ac component => Cannot penetrate the wall of the vacuum chamber, as it was the case for the electric images)

=> Boundary condition on the plates:

$$B_{\perp} = 0$$



BEAM OFF-AXIS BETWEEN 2 INFINITE PC // PLATES (6/12)



=> $\boxed{B_y = 0}$ on the upper plate, as it should be for a PC wall

=> This image (going OUT of the paper) will have another image (going IN the paper) for the lower plate (and so on)

=> Finally, the same thing applies for the interaction between the beam and the lower plate

BEAM OFF-AXIS BETWEEN 2 INFINITE PC // PLATES (7/12)

- **The picture is very similar to the case of the electric images, except for some change in the direction of each force component**

=> $$\frac{F_y^{mag,ac}}{e} = -\,\beta^2 \, E_y$$

=> Gathering both results (from electric and magnetic ac images), one therefore obtains

$$\frac{F_y^{ele+mag,ac}}{e} = \frac{E_y}{\gamma^2}$$

BEAM OFF-AXIS BETWEEN 2 INFINITE PC // PLATES (8/12)

- **The image forces acting on the witness particle y_1 come directly from the individual images, therefore the electric field and magnetic flux density from the images at the location of the witness particle satisfy source-free electric (and magnetic) Gauss's laws**

$$\boxed{div\ \vec{E}_i = 0}$$

=> $$\frac{\partial E_{ix}}{\partial x} + \frac{\partial E_{iy}}{\partial y} = 0$$ **=>** $$\boxed{\varepsilon_{1x} = -\,\varepsilon_{1y}}$$

- **Furthermore, from translational invariance in the case of 2 infinite horizontal plates, one needs to have** $\boxed{\xi_{1x} = 0}$

=> $$\boxed{E_x = \frac{\lambda}{2\,\pi\,\varepsilon_0\,h^2}\left(\frac{\pi^2}{24}\,\bar{x}_1 - \frac{\pi^2}{24}\,x_1\right)}$$

- **For a cylindrical beam pipe, one thus has** $\varepsilon_{1x} = -\,\varepsilon_{1y} = 0$ **(because of the symmetry between horizontal and vertical)**

BEAM OFF-AXIS BETWEEN 2 INFINITE PC // PLATES (9/12)

- **Comparison between 2 infinite PC // plates and a circular PC beam pipe, in the case of ac magnetic fields**

✧ **Circular PC beam pipe (radius b)**

$$\boxed{F_x = \Lambda_c \; \bar{x}_1}$$

$$\boxed{F_y = \Lambda_c \; \bar{y}_1}$$

with $$\boxed{\Lambda_c = \frac{\lambda \, e}{2 \, \pi \, \varepsilon_0 \, \gamma^2 \, b^2}}$$

✧ **2 infinite PC (horizontal) // plates (1/2 gap = h = b)**

$$\boxed{F_x = \Lambda_c \left(\frac{\pi^2}{24} \, \bar{x}_1 - \frac{\pi^2}{24} \, x_1 \right)}$$

$$\boxed{F_y = \Lambda_c \left(\frac{\pi^2}{12} \, \bar{y}_1 + \frac{\pi^2}{24} \, y_1 \right)}$$

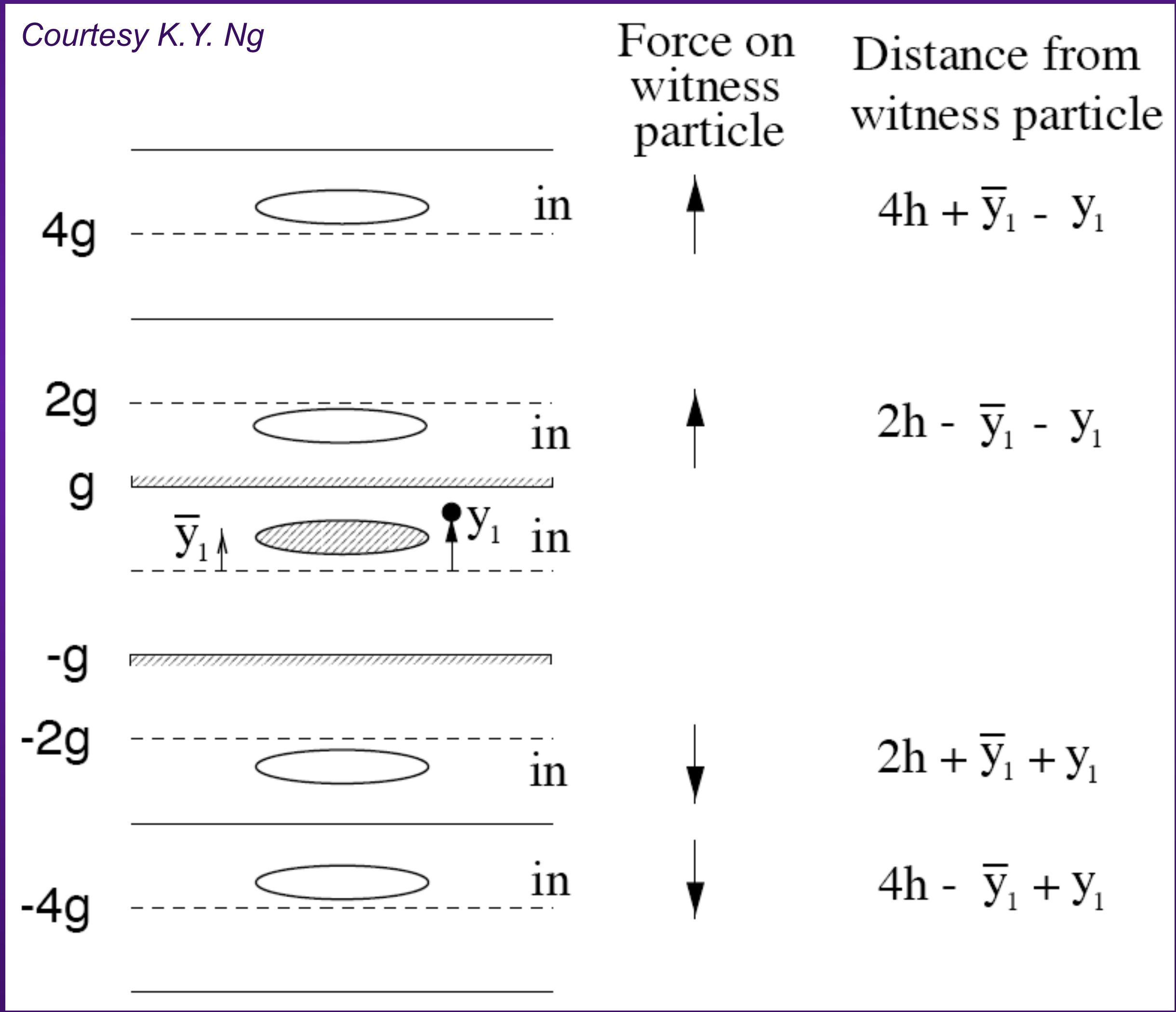
BEAM OFF-AXIS BETWEEN 2 INFINITE PC // PLATES (10/12)

- **MAGNETIC IMAGES (2)**

  (dc component => Can penetrate the wall of the vacuum chamber and land on the pole faces of the magnet)

=> Boundary condition on the magnet poles:

$$B_t = 0$$



BEAM OFF-AXIS BETWEEN 2 INFINITE PC // PLATES (11/12)

- To achieve this we need images such that all the image currents flow in exactly the same direction as the beam (i.e. all IN)

=>
$$\frac{F_y^{mag,dc}}{e} = \beta^2 \frac{\lambda}{2\,\pi\,\varepsilon_0} \left[\begin{array}{l} + \dfrac{1}{2\,g - \overline{y}_1 - y_1} - \dfrac{1}{2\,g + \overline{y}_1 + y_1} + \dfrac{1}{6\,h - \overline{y}_1 - y_1} - \dfrac{1}{6\,h + \overline{y}_1 + y_1} + \ldots \\ + \dfrac{1}{4\,g + \overline{y}_1 - y_1} - \dfrac{1}{4\,g - \overline{y}_1 + y_1} + \dfrac{1}{8\,h + \overline{y}_1 - y_1} - \dfrac{1}{8\,h - \overline{y}_1 + y_1} + \ldots \end{array} \right]$$

=>
$$\boxed{\frac{F_y^{mag,dc}}{e} = \frac{\lambda\,\beta^2}{2\pi\,\varepsilon_0\,g^2} \left(\frac{\pi^2}{24}\,\overline{y}_1 + \frac{\pi^2}{12}\,y_1 \right)}$$

One can define the Laslett coherent dc magnetic image coefficient ξ_{2y} (obtained when $\overline{y}_1 = y_1$), and $2\xi_{2y} = \pi^2 / 8$

It is called $2\varepsilon_{2y}$ in the literature = Laslett incoherent dc magnetic image coefficient

BEAM OFF-AXIS BETWEEN 2 INFINITE PC // PLATES (12/12)

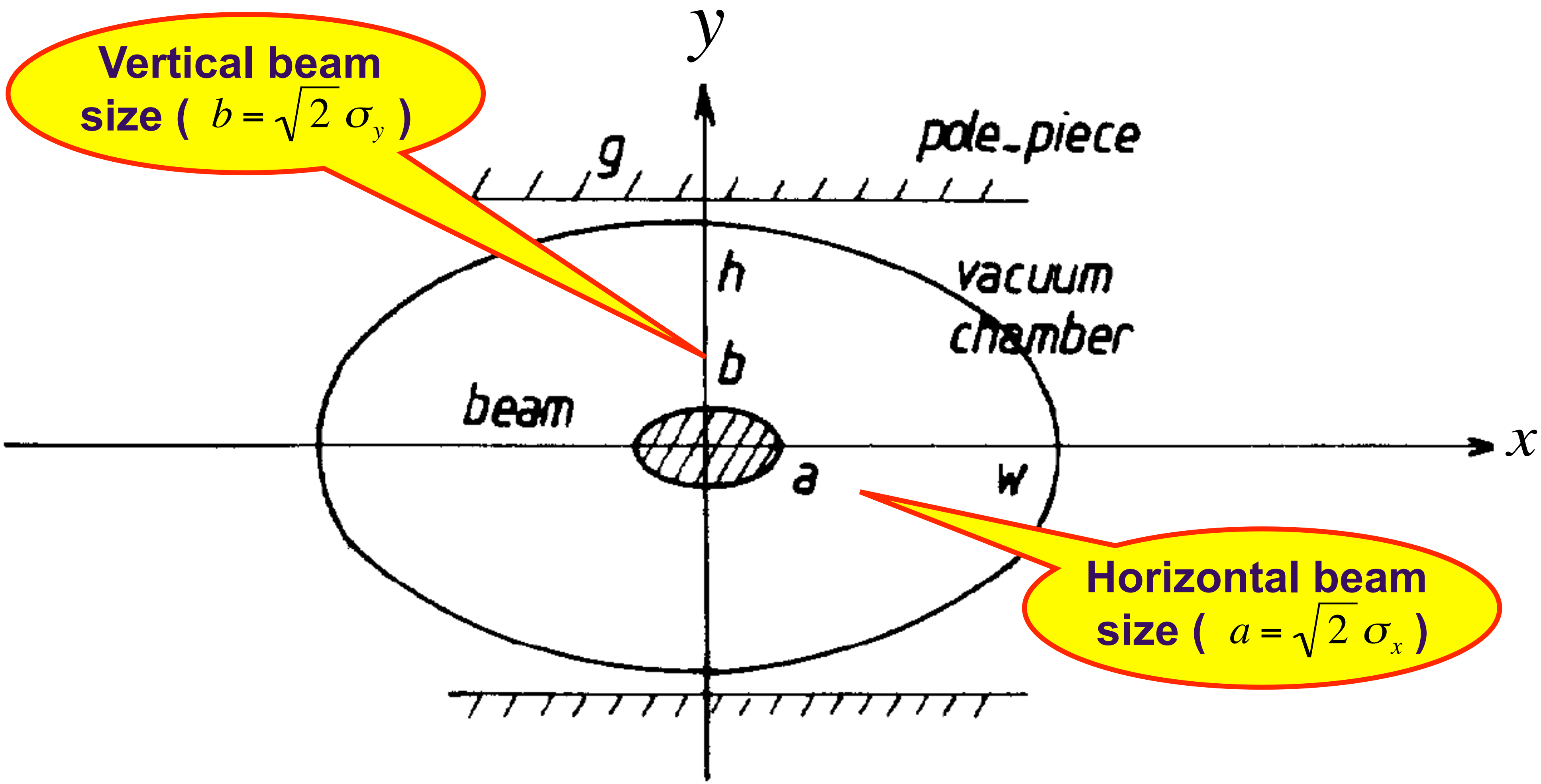
- Similarly, $\boxed{\varepsilon_{2x} = -\,\varepsilon_{2y}}$

- Furthermore, from translational invariance in the case of 2 infinite horizontal plates, one needs to have $\boxed{\xi_{2x} = 0}$

$$\Rightarrow \quad \boxed{\frac{F_x^{mag,dc}}{e} = \frac{\lambda\,\beta^2}{2\pi\,\varepsilon_0\,g^2}\left(\frac{\pi^2}{12}\,\bar{x}_1 - \frac{\pi^2}{12}\,x_1\right)}$$

- For a cylindrical beam pipe, one thus has $\varepsilon_{2x} = -\,\varepsilon_{2y} = 0$ (because of the symmetry between horizontal and vertical)

GENERAL FORMULAE FOR THE TUNE SHITFS (1/9)



GENERAL FORMULAE FOR THE TUNE SHITFS (2/9)

Laslett coefficients	Circular ($a = b,\ w = h$)	Elliptical (e.g. $w = 2h$)	Parallel plates ($h/w = 0$)
ε_0^x	$1/2$	$\frac{b^2}{a(a+b)}$	
ε_0^y	$1/2$	$\frac{b}{a+b}$	
ε_1^x	0	-0.172	-0.206
ε_1^y	0	0.172	0.206
ξ_1^x	$1/2$	0.083	0
ξ_1^y	$1/2$	0.55	$0.617(\pi^2/16)$
ε_2^x	$-0.411(-\pi^2/24)$	-0.411	-0.411
ε_2^y	$0.411(\pi^2/24)$	0.411	0.411
ξ_2^x	0	0	0
ξ_2^y	$0.617(\pi^2/16)$	0.617	0.617

GENERAL FORMULAE FOR THE TUNE SHITFS: COASTING BEAMS (3/9)

- **Single-particle equation of motion (e.g. in the vertical plane)**

$$\ddot{y} + Q_{y0}^2 \, \Omega_0^2 \; y = \frac{F_y}{\gamma \, m_0}$$

$$F_y^{\,pert}$$

- **The (perturbative) force can be expanded to first order in terms of the test particle's motion and the average beam position to give**

$$F_y = \left(\frac{\partial F_y}{\partial y} \right)_{\bar{y}=0} y + \left(\frac{\partial F_y}{\partial \bar{y}} \right)_{y=0} \bar{y}$$

This beam force is therefore static (or dc)

=> $Q_y^2 \approx Q_{y0}^2 + 2 \, Q_{y0} \, \Delta Q_{incoh}^y$ **with** $\Delta Q_{incoh}^y = - \dfrac{1}{2 \, Q_{y0} \, \Omega_0^2 \, \gamma \, m_0} \left(\dfrac{\partial F_y}{\partial y} \right)_{\bar{y}=0}$

GENERAL FORMULAE FOR THE TUNE SHITFS: COASTING BEAMS (4/9)

Averaged around the machines

$$\Delta Q^{y}_{incoh} = - \frac{N\,R\,r_p}{\pi\,Q_y\,\gamma\,\beta^2} \left(< \frac{\varepsilon_1^y}{h^2} > + \beta^2 < \frac{\varepsilon_2^y}{g^2} > + \frac{\varepsilon_0^y}{\gamma^2\,b^2} \right)$$

Electric image in vacuum chamber

Magnetic image in magnet poles

Self field

- The coherent motion can be solved by choosing $\boxed{y = \bar{y}}$

$$\Delta Q^{y}_{coh} = - \frac{1}{2\,Q_{y0}\,\Omega_0^2\,\gamma\,m_0} \left[\left(\frac{\partial F_y}{\partial y} \right)_{\bar{y}=0} + \left(\frac{\partial F_y}{\partial \bar{y}} \right)_{y=0} \right]$$

GENERAL FORMULAE FOR THE TUNE SHITFS: COASTING BEAMS (5/9)

- The only ac magnetic field comes from betatron oscillations
  - Low frequency when the betatron tune is close to an integer => Penetrating magnetic field
  - High frequency when the betatron tune is close to a 1/2 integer => Non-penetrating magnetic field

$$f_y = \left(n_y + Q_c \right) f_0$$

See "coasting beam instabilities"

Azimuthal mode number

Coherent tune

- Penetrating magnetic field

$$\Delta Q^y_{coh\,,dc\,mag} = - \frac{N\,R\,r_p}{\pi\,Q_y\,\gamma\,\beta^2} \left(< \frac{\xi_1^y}{h^2} > + \beta^2 < \frac{\xi_2^y}{g^2} > \right)$$

Electric image in vacuum chamber

Magnetic image in magnet poles

GENERAL FORMULAE FOR THE TUNE SHITFS: COASTING BEAMS (6/9)

- **Non penetrating magnetic field**

$$\Delta Q^{y}_{coh\,,ac\,mag} = - \frac{N\,R\,r_p}{\pi\,Q_y\,\gamma\,\beta^2} \left(< \frac{\xi_1^y}{h^2} > + \beta^2 < \frac{\varepsilon_2^y}{g^2} > - \beta^2 < \frac{\xi_1^y - \varepsilon_1^y}{h^2} > \right)$$

Electric image in vacuum chamber

Magnetic image in magnet poles

ac magnetic image in vacuum chamber

=>

$$\boxed{\Delta Q^{y}_{coh\,,ac\,mag} = - \frac{N\,R\,r_p}{\pi\,Q_y\,\gamma\,\beta^2} \left[< \frac{\xi_1^y}{\gamma^2\,h^2} > + \beta^2 \left(< \frac{\varepsilon_1^y}{h^2} > + < \frac{\varepsilon_2^y}{g^2} > \right) \right]}$$

GENERAL FORMULAE FOR THE TUNE SHITFS: BUNCHED BEAMS (7/9)

- The ac magnetic field now comes from 2 sources
  - Transverse betatron oscillation of the bunch (as before)
  - Longitudinal (or axial) bunching of the beam $f_{RF} = h\ f_0$

=> We assume that it is always ac (high frequency)

There is no *B* here because they are dc fields coming from the average beam current

$$\Delta Q_{incoh}^{y} = - \frac{N\,R\,r_p}{\pi\,Q_y\,\gamma\,\beta^2} \left(\frac{1}{B} < \frac{\varepsilon_1^y}{h^2} > + \beta^2 < \frac{\varepsilon_2^y}{g^2} > - \beta^2 \left(\frac{1}{B} - 1 \right) < \frac{\varepsilon_1^y}{h^2} > + \frac{\varepsilon_0^y}{B\,\gamma^2\,b^2} \right)$$

Electric image in vacuum chamber

Magnetic image in magnet poles

ac magnetic image from axial bunching

Self field

GENERAL FORMULAE FOR THE TUNE SHITFS: BUNCHED BEAMS (8/9)

=> $$\Delta Q^{y}_{incoh} = -\frac{N\, R\, r_p}{\pi\, Q_y\, \gamma\, \beta^2}\left[\left(\frac{1}{B\,\gamma^2} + \beta^2\right)\left\langle \frac{\varepsilon_1^y}{h^2}\right\rangle + \beta^2 \left\langle \frac{\varepsilon_2^y}{g^2}\right\rangle + \frac{\varepsilon_0^y}{B\,\gamma^2\, b^2}\right]$$

- **Penetrating magnetic field**

$$\Delta Q^{y}_{coh\,,dc\,mag} = -\frac{N\, R\, r_p}{\pi\, Q_y\, \gamma\, \beta^2}\left[\frac{1}{B}\left\langle \frac{\xi_1^y}{h^2}\right\rangle + \beta^2 \left\langle \frac{\xi_2^y}{g^2}\right\rangle - \beta^2 \left(\frac{1}{B} - 1\right)\left\langle \frac{\xi_1^y}{h^2}\right\rangle\right]$$

Electric image in vacuum chamber

Magnetic image in magnet poles

ac magnetic image from axial bunching

=> $$\Delta Q^{y}_{coh\,,dc\,mag} = -\frac{N\, R\, r_p}{\pi\, Q_y\, \gamma\, \beta^2}\left[\left(\frac{1}{B\,\gamma^2} + \beta^2\right)\left\langle \frac{\xi_1^y}{h^2}\right\rangle + \beta^2 \left\langle \frac{\xi_2^y}{g^2}\right\rangle\right]$$

GENERAL FORMULAE FOR THE TUNE SHITFS: BUNCHED BEAMS (9/9)

- **Non penetrating magnetic field**

$$\Delta Q^{y}_{coh\,,ac\,mag} = -\frac{N\,R\,r_p}{\pi\,Q_y\,\gamma\,\beta^2}\left[\frac{1}{B} < \frac{\xi_1^y}{h^2} > + \beta^2 < \frac{\varepsilon_2^y}{g^2} > - \beta^2 < \frac{\xi_1^y - \varepsilon_1^y}{h^2} > - \beta^2\left(\frac{1}{B} - 1\right) < \frac{\xi_1^y}{h^2} >\right]$$

ac magnetic image from axial bunching

Electric image in vacuum chamber

Magnetic image in magnet poles

ac magnetic image from transverse motion

=>

$$\boxed{\Delta Q^{y}_{coh\,,ac\,mag} = -\frac{N\,R\,r_p}{\pi\,Q_y\,\gamma\,\beta^2}\left[\frac{1}{B\,\gamma^2} < \frac{\xi_1^y}{h^2} > + \beta^2\left(< \frac{\varepsilon_1^y}{h^2} > + < \frac{\varepsilon_2^y}{g^2} > \right)\right]}$$

A PRACTICAL FORMULA FOR THE MAXIMUM TRANSVERSE INCOHERENT DIRECT SC TUNE SHIFT

$$\Delta Q_{incoh}^{y0} = -\frac{N R r_p}{\pi Q_y \gamma \beta^2} \frac{1}{B \gamma^2} \frac{\varepsilon_0^y}{b^2}$$

$$\varepsilon_0^y = \frac{b}{a+b}$$

=>

$$\boxed{\Delta Q_{incoh}^{y0} = -\frac{2 r_p I_p < \beta_y > R}{e c \beta^3 \gamma^3 a \left(a + b \right)}}$$

$I_p = 3 e N_b / 2 \tau_b$ **Bunch peak current considering a parabolic line density**

$L_b = \beta c \tau_b$ $\quad < \beta_y > \approx R / Q_y$ $\quad a = \sqrt{2} \sigma_x$ $\quad b = \sqrt{2} \sigma_y$

TRANSVERSE INCOHERENT DIRECT SC TUNE SHIFT FORMULA FOR IONS

Charge state of the ion

$$\Delta Q_{incoh}\left(\text{ion}\right)=\Delta Q_{incoh}\left(\text{proton}\right)\times\frac{Z^{2}}{A}$$

Mass number of the ion