Collective effects in the longitudinal plane

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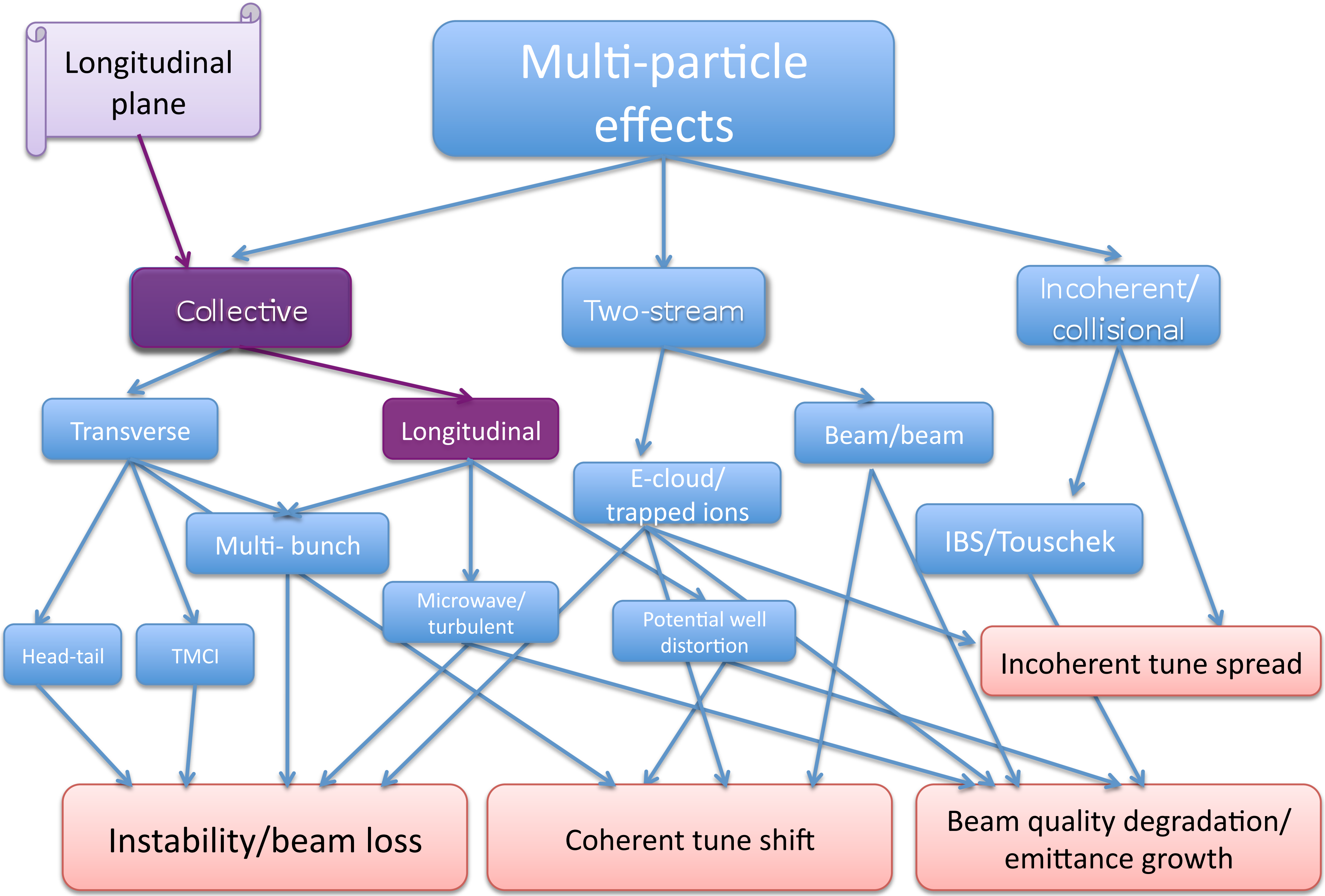
USPAS Course on collective effects

Tuesday, 23.06.2009


Longitudinal plane
Multi-particle effects
Collective
Two-stream
Incoherent/ collisional
Transverse
Longitudinal
Beam/beam
E-cloud/ trapped ions
IBS/Touschek
Multi- bunch
Microwave/ turbulent
Potential well distortion
Head-tail
TMCI
Incoherent tune spread
Instability/beam loss
Coherent tune shift
Beam quality degradation/ emittance growth

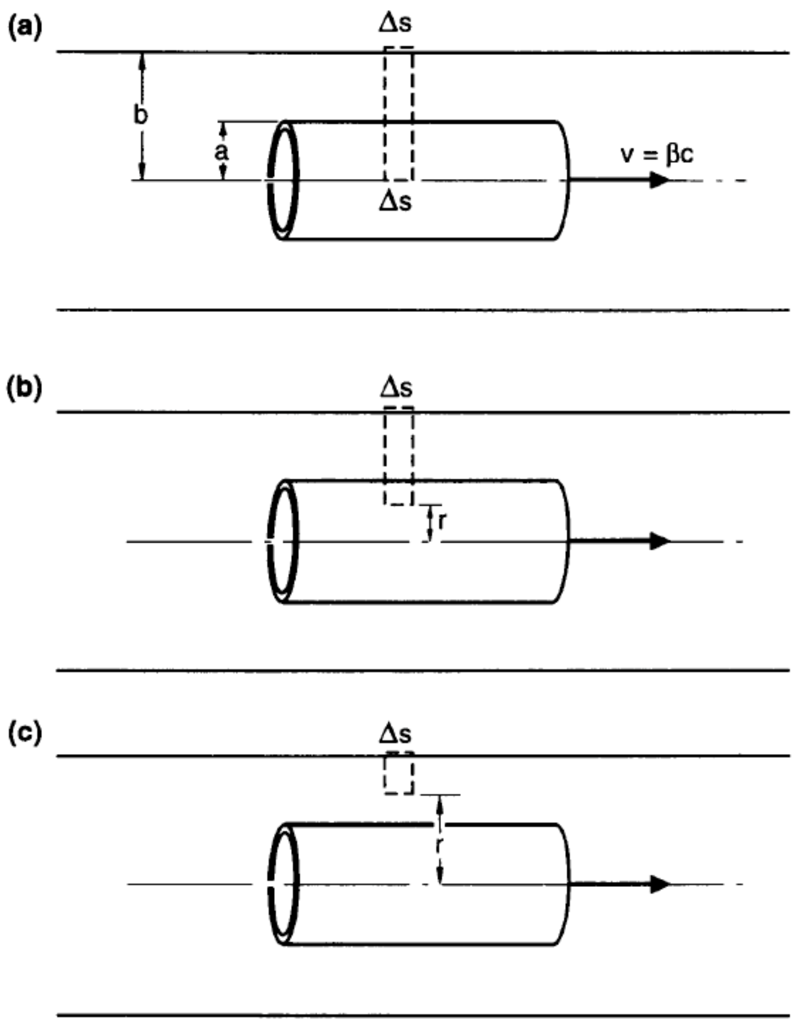
Program of the day

- Space charge in the longitudinal plane
- Equations of motion in the longitudinal plane
  - Stationary bucket
  - Accelerating bucket
- Synchrotron tune shift due to space charge
- Energy loss (single pass, multi pass)
- Vlasov equation:
  - Stationary distributions without collective terms (linear and nonlinear matching)
  - Stationary distributions with collective terms (potential well distortion)
    - Haissinski equation
    - Synchronous phase shift, bunch lengthening or shortening, synchrotron tune shift
  - Non-stationary solutions: perturbative approach
    - Bunched beams (azimuthal and radial modes for low beam intensities, mode coupling, turbulent bunch lengthening)
    - Coasting beams

Some references (reading recommended)

- Yesterday's lecture on Space Charge, Wake Fields and Impedances (E. Métral)
- A. Chao, "Physics of collective beam instabilities in high energy accelerators"
  - Chapter 1, Introduction, pages 20-27
  - Chapter 2, Wake fields and impedances, pages 117-126
  - Chapter 6, Perturbation formalism, pages 273-332, 361-363
  - Chapter 5, Landau damping, pages 251-263
  - Chapter 4, Macroparticle models, pages 162-172

Space charge



We do the calculation for a ring-shaped distribution $\lambda(s\text{-}\beta ct)$

$$E_r = \begin{cases} 0 & \text{if} \quad r < a \\ \dfrac{\lambda e}{2\pi\epsilon_0 r} & \text{if} \quad a < r < b \end{cases}$$

$$B_\theta = \frac{\beta E_r}{c}$$

3rd Maxwell equation, or Faraday-Neumann law

$$\oint_{\partial\Omega} \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\int_{\Omega} \vec{B} \cdot d\vec{S}$$

$$E_s(s,r)\Delta s + \int_r^b E_r(r',s)dr' + \int_b^r E_r(r', s+\Delta s)dr' = -\frac{d}{dt}\int_r^b B_\theta(r',s)dr'\Delta s$$

Space charge

- All dependencies on s are actually on *s-βct*
- Therefore *d/dt =-βc d/ds*
- We carry out the integrals in the equation obtained and take the limit for *Δs->0*

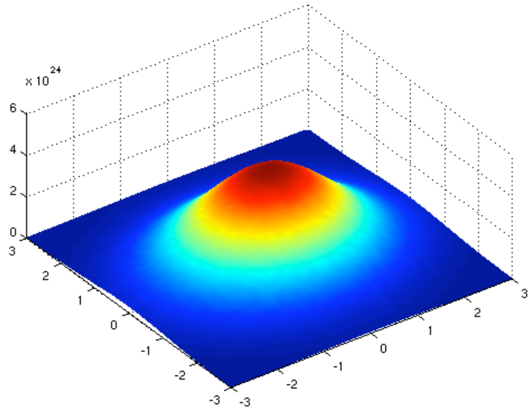
$$E_s(s,r)\Delta s+\int_r^b E_r(r',s)dr'+\int_b^r E_r(r',s+\Delta s)dr'=-\frac{d}{dt}\int_r^b B_\theta(r',s)dr'\Delta s$$

$$E_s(s,t,r)=\begin{cases}-\dfrac{e}{2\pi\epsilon_0\gamma^2}\lambda'(s-\beta ct)\log\dfrac{b}{a} & \text{if} \quad 0<r<a\\ -\dfrac{e}{2\pi\epsilon_0\gamma^2}\lambda'(s-\beta ct)\log\dfrac{b}{r} & \text{if} \quad a<r<b\end{cases}$$

Space charge

- We have found the space charge longitudinal electric field for a cylindrically symmetric ring shaped distribution
- Let's assume now that we have a beam with a radial distribution n(r)
- Then for the linearity of Maxwell's equations, we can calculate its longitudinal space charge electric field as the superposition of the contributions given by the various rings composing the actual distribution

$$\int_0^\infty 2\pi r n(r) dr = 1$$



$$E_s(r, s-\beta ct) = -\frac{e}{2\pi\epsilon_0\gamma^2}\lambda'(s-\beta ct)\left[\log\frac{b}{r} + \int_r^b 2\pi r' n(r') \log\frac{r}{r'} dr'\right]$$

Space charge

$$E_s(r, s-\beta ct) = -\frac{e}{2\pi\epsilon_0\gamma^2}\lambda'(s-\beta ct)\left[\log\frac{b}{r} + \int_r^b 2\pi r' n(r')\log\frac{r}{r'}dr'\right]$$

- Longitudinal space charge has the following interesting dependencies:
  - It decreases with energy like γ^{-2}. Therefore it vanishes in the ultrarelativistic limit
  - It is proportional to the opposite of the derivative of the line density $-\lambda'$. This can be understood intuitively because it must be directed from a region with higher charge density to a region with lower charge density (i.e. it pushes with the opposite of the gradient of the line charge)
  - Space charge would then spread out charge bumps. However, remember that only below transition energy, accelerated particles go faster and space charge has this smoothing action. Above transition, accelerated particles take a longer time to go around the accelerator and density peaks can be enhanced. This is the origin of the so-called **negative mass instability**. Momentum spread (unbunched beams) or synchrotron motion (bunched beams) can usually stabilize this effect.

Space charge: exercises

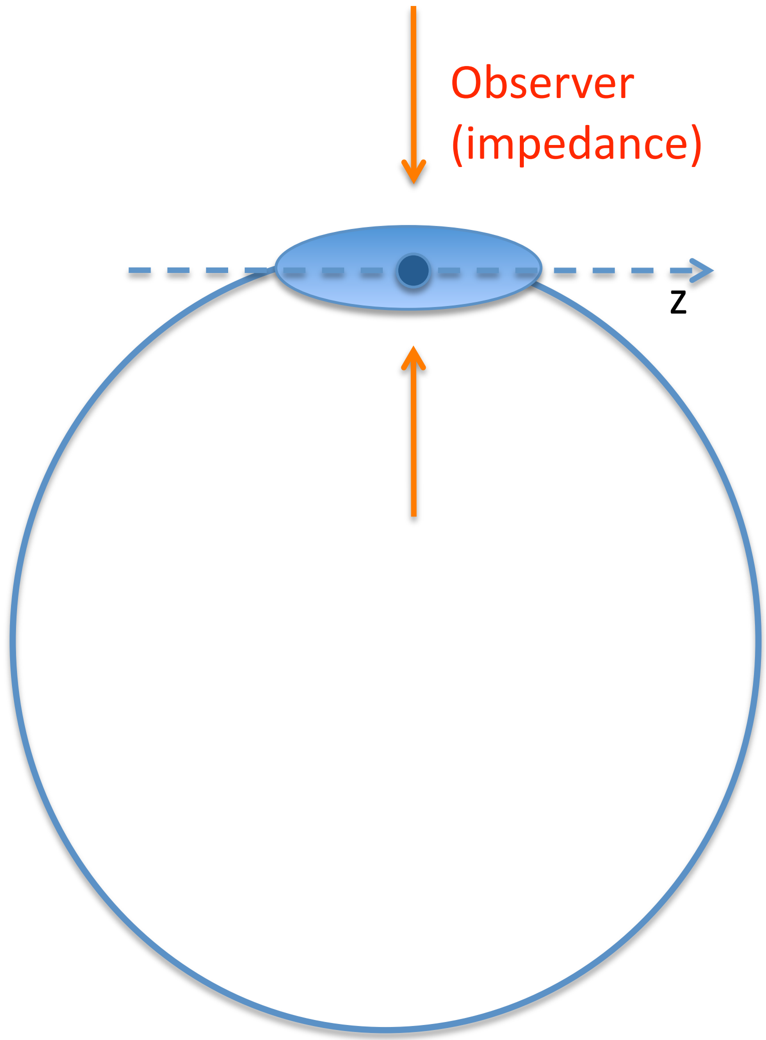
1. Prove the formula on the previous page (Slide 8)
2. Calculate the radial distribution of E_s for a bunch with a uniform distribution for $r<a$ and with parabolic radial distribution $n(r)=k(a^2-r^2)$ for $r<a$
3. Verify that the fields on Slide 5 and 6 (E_r, B_θ, E_s) do not satisfy Maxwell's equations, unless a correction term is taken into account. Calculate the leading term for the correction and show that it can be neglected as long as the bunch is much longer than b/γ
4. Compare the strength of transverse and longitudinal space charge forces, and find under which condition the transverse one is dominating.

$$\int x^3 \log x dx = -\frac{x^4}{16} + \frac{1}{4} x^4 \log x$$

$$\frac{1}{r}\frac{\partial(rB_\theta)}{\partial r} - \frac{1}{r}\frac{\partial B_r}{\partial \theta} - \mu_0\epsilon_0 \frac{\partial E_s}{\partial t} = \mu_0 j_s$$

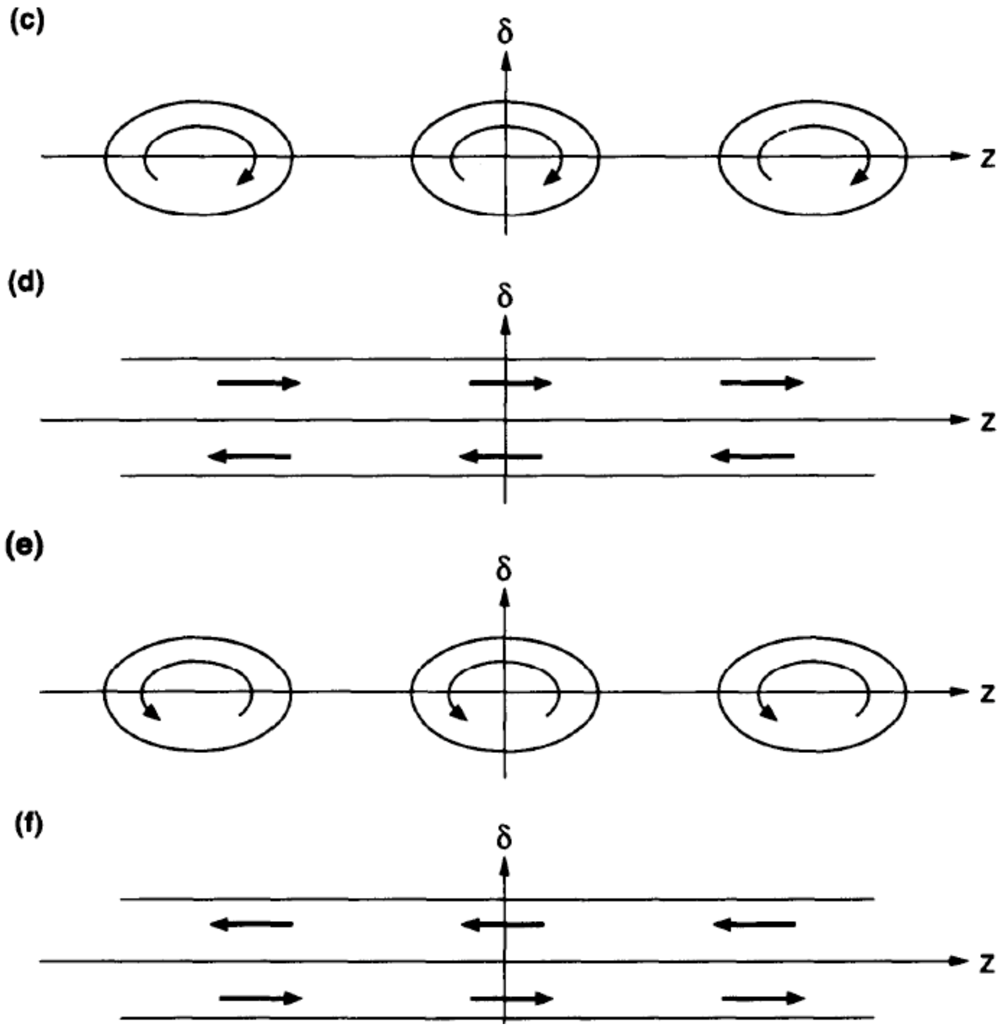
Equations of longitudinal dynamics

- To describe the motion of particles in the longitudinal phase space, we will use the pair **(z,δ)**, which represent the arrival delay of the particle at an observation point (multiplied by the particle velocity, $z=-\beta c\tau$) and its relative momentum offset, respectively, with respect to the synchronous particle.
- The pair is s-dependent, i.e. it is defined with respect to a chosen observation point in the accelerator, but this is not critical for the longitudinal plane, because longitudinal motion is generally much slower than the revolution time
- $(z,\delta)=(0,0)$ for the synchronous particle



Equations of longitudinal dynamics

- The synchronous particle has zero momentum offset and always takes T_0 to go around, i.e. is always observed with z=0.
- Particles with positive z arrive earlier at the observer (negative delay τ), those with negative z arrive later (positive time delay τ)
- Bunched beams: below transition particles are focused back by deceleration. Opposite above transition.
- Coasting beams: below transition particles with positive momentum offset shear toward positive z (absence of focusing). Opposite above transition



Equations of longitudinal dynamics

$$T = \frac{C}{\beta c} \qquad \frac{\Delta C}{C_0} = \alpha\delta = \delta \oint_C \frac{D(s)}{\rho} ds$$

$$\frac{\Delta T}{T_0} = \frac{\Delta C}{C_0} - \frac{\Delta \beta}{\beta_0} \qquad \delta = \frac{p - p_0}{p_0}$$

The difference in revolution time between a generic particle and the synchronous particle, that is its delay in arrival time at a point, can be easily related to its momentum offset δ

$$\frac{\Delta T}{T_0} = \left(\alpha - \frac{1}{\gamma^2}\right)\delta = \eta\delta$$

Equations of longitudinal dynamics

Reminder of vocabulary and meaning of symbols:

α is the momentum compaction factor
η is the slippage factor
C is the mean circumference of the accelerator C=2πR
γ_t is the γ transition (corresponding to the transition energy)

$$\alpha = \frac{1}{\gamma_t^2}$$

Remember that γ is a property of the beam, whereas γ_t is a property of the machine. η depends on both, but in the limit of very high energies, it tends to α, and therefore depends only on the lattice.

$$\eta = \alpha - \frac{1}{\gamma^2} = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} \Rightarrow \eta \begin{cases} < 0 & \text{if } \gamma < \gamma_t \\ > 0 & \text{if } \gamma > \gamma_t \end{cases}$$

Equations of longitudinal dynamics

$$\frac{-\beta_0 c \Delta T}{T_0} = \frac{\Delta z}{T_0} \approx \boxed{\frac{dz}{dt} = -\beta_0 c \eta \delta}$$

Energy change per turn

$$\Delta E_{cavity}(z) = eV_{cav}(z)$$

Actually a kick that the beam receives at the cavity location(s)

$$\Delta E_{impedance}(z) = eV_{imp}(z)$$

Actually a kick that the beam receives at the impedance location(s)

$$\Delta E_{sc}(z) = eCE_{sc}(z)$$

Actually a force felt by the beam constantly over the revolution

$$\Delta E_{tot} = e[V_{cav}(z) + V_{imp}(z) + CE_{sc}(z)] = eV_{tot}(z)$$

$$\frac{\Delta p}{T_0} \approx \frac{dp}{dt} = \frac{eV_{tot}(z)}{\beta_0 c T_0} = \frac{eV_{tot}(z)}{C}$$

$$\boxed{\frac{d\delta}{dt} = \frac{eV_{tot}(z)}{p_0 C}}$$

Equations of longitudinal dynamics

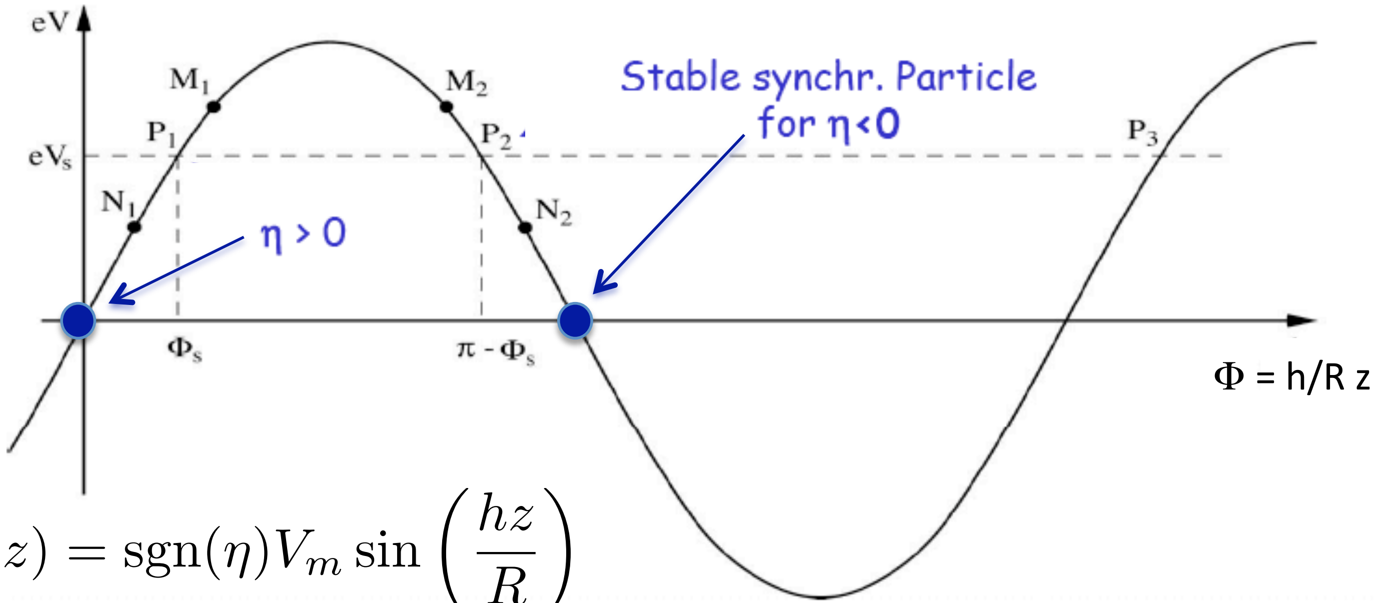
$$\begin{cases} \dfrac{dz}{dt} = -\eta\beta_0 c\delta \\ \dfrac{d\delta}{dt} = \dfrac{eV(z)}{p_0 C} \end{cases}$$

V(z) is the sum of the external voltage from the rf systems and self-induced from space charge and impedances. The rf component is 0 when z=0 for a stationary bucket.
Above transition, the regime is negative-mass due to $\eta>0$: the particle will move to lower z for positive momentum deviation, and vice versa.
We are interested in the particle motion only in the stable regions, i.e. where particle motion is confined around a stable point. These regions are called **rf buckets** and are determined by the phase of the rf with the motion of the synchronous particle

Equations of longitudinal dynamics

Example: *V(z)* is only external due to the main rf system and has a sinusoidal waveform tuned on harmonic number *h* (means that the rf frequency ω_{rf} is *h* times the revolution frequency ω_0). Case of **stationary bucket**

$$V(z) = V_m \sin\left(\frac{hz}{R} + \Phi_s\right) \quad \text{with} \quad \Phi_s = \begin{cases} 0 & \text{above transition} \\ \pi & \text{below transition} \end{cases}$$



$$V(z) = \text{sgn}(\eta) V_m \sin\left(\frac{hz}{R}\right)$$

Equations of longitudinal dynamics

If *V(z)* is can be derived from a scalar potential U(z), the system is Hamiltonian and we can also express its Hamiltonian function.

$$V(z) = -\frac{dU}{dz} \Rightarrow \boxed{H = -\frac{1}{2}\beta c \eta \delta^2 + \frac{e}{p_0 C} U(z)}$$

$$\begin{cases} \dot{z} = \dfrac{\partial H}{\partial \delta} \\ \dot{\delta} = -\dfrac{\partial H}{\partial z} \end{cases}$$

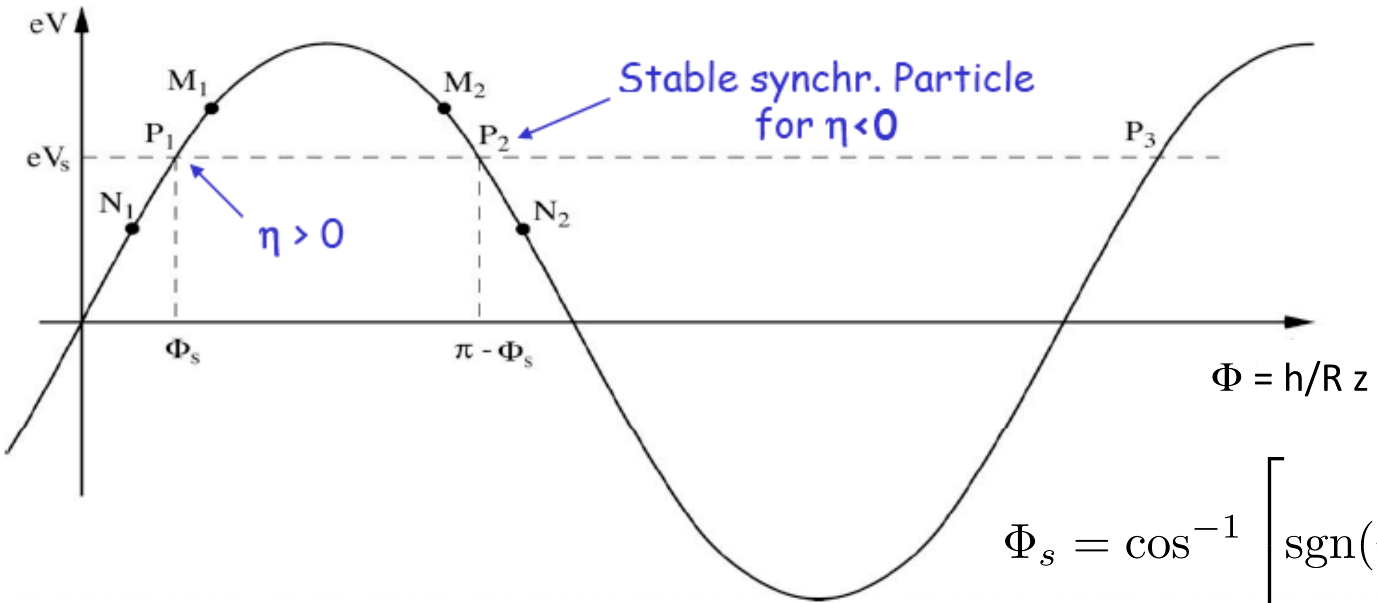
$$\frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{\delta}}{\partial \delta} = 0$$

$$\frac{\partial H}{\partial t} = 0$$

Equations of longitudinal dynamics

For an **accelerating bucket**, the bunch needs to be synchronized around a phase different from 0 or π, which can guarantee a desired rate of momentum (energy) increase per turn. Obviously, the rf system can only accelerate up to an energy gain per turn [eV] that equals its maximum voltage [V] (ideally, because the bucket area shrinks to zero when $V_m = \Delta E_{acc}$).

$$\sin \Phi_s = \frac{C}{eV_m} \left[\frac{\Delta p_0}{T_0} \right]_{acc} = \frac{\Delta E_{acc}}{eV_m} \;\Rightarrow V_m[\mathrm{V}] > \Delta E_{acc}[\mathrm{eV}]$$



$$\Phi_s = \cos^{-1} \left[\mathrm{sgn}(\eta) \sqrt{1 - \left(\frac{\Delta E_{acc}}{eV_m} \right)^2} \right]$$

Equations of longitudinal dynamics

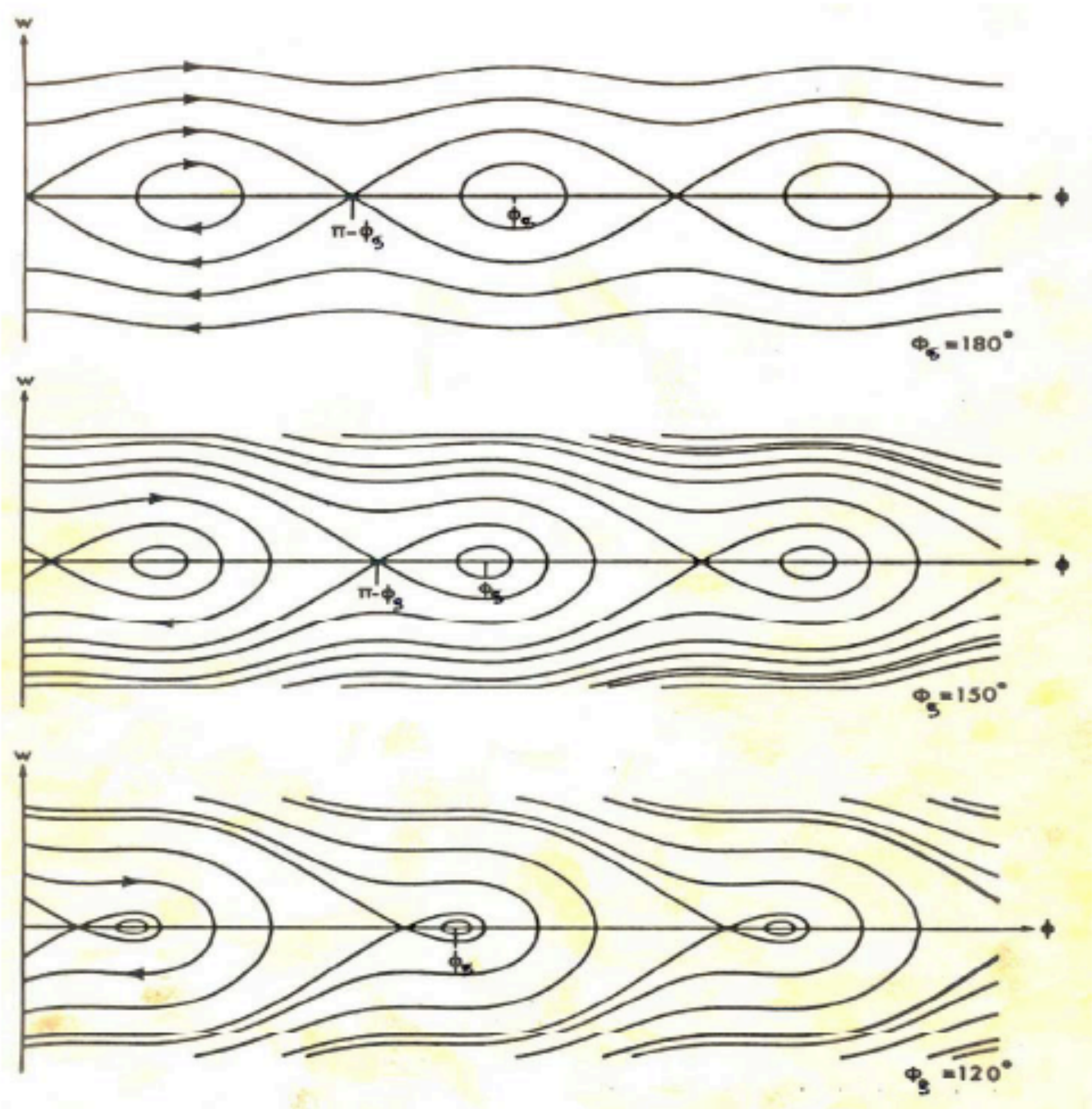
For an **accelerating bucket**, we redefine the pair of coordinates we use to describe the system, in such a way that they keep being both zero for the synchronous particle The acceleration rate does not need to be constant in time….

$$\Delta p = p - p_0(t) \;\; \text{and} \;\; \zeta = z - \frac{C}{2\pi h}\Phi_s(t)$$

$$\begin{cases} \dfrac{d\zeta}{dt} = -\eta(t)\beta(t)c\dfrac{\Delta p}{p_0(t)} \\ \dfrac{d\Delta p}{dt} = \dfrac{eV_m}{C}\left[\sin\left(\dfrac{h\zeta}{R} + \Phi_s(t)\right) - \sin\Phi_s(t)\right] \end{cases}$$

Usually Φ_s is not constant at the beginning and end of an accelerating ramp, to make the curve B(t) smoother (it is made linear piece-wise). Also when crossing transition, there might be a γ_t-jump scheme in place to minimize the time the beam spends on transition. The above equations can be derived by a slowly time varying Hamiltonian H(t).

Equations of longitudinal dynamics



As the synchronous phase gets closer to 90° the area of stable motion (closed trajectories) gets smaller. These areas are often called "BUCKET".

The number of circulating buckets is equal to "h".

The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which correspond to no acceleration . The RF acceptance increases with the RF voltage.

Equations of longitudinal dynamics

- Usually p_0, Φ_s, β, γ, etc. vary on a time scale that is long with respect to the time the synchrotron motion, described in the longitudinal phase space (ζ, Δp), evolves. Yet another time scale in the beam particle dynamics inside an accelerator:
  - Transverse motion has a periodicity of fractions of a turn
  - Longitudinal motion has typically a periodicity of ~100 turns (only when crossing transition longitudinal motion becomes much slower)
  - Acceleration takes usually several hundreds of thousands of turns.
- Even if the beam is usually accelerated at the expense of one (the main) rf system alone, the longitudinal dynamics can make use of more than one rf-system. Additional rf systems are usually programmed to be 'slaves' of the main system, i.e. they are made to have their 0 or π phase correspondingly to the synchronous phase, which is solely determined by the main rf.
  - For example the CERN-PSB accelerates the beam over the full length of its cycle and makes use of the 2 rf systems h=1 and h=2. In some configurations, h=1 is the main rf system and h=2 is used for flattening the bunch against space charge. In some other configurations, h=2 is the main system and h=1 is used to reduce the bunch spacing and synchronize the transfer into the CERN-PS waiting buckets.

Equations of longitudinal dynamics

Some times the bunch is made to sit in an **accelerating bucket** only to compensate for external losses

- in a lepton storage ring, to compensate for synchrotron radiation losses
- in general, a bunch in a stationary bucket can move to a synchronous phase different from 0 or π in order to compensate for impedance losses (see further)

$$\begin{cases} \dfrac{d\zeta}{dt} = -\eta\beta c\delta \\ \dfrac{d\delta}{dt} = \dfrac{eV_m}{p_0 C}\left[\sin\left(\dfrac{h\zeta}{R} + \Phi_s\right) - \sin\Phi_s\right] \end{cases}$$

$$H = -\frac{1}{2}\beta c\eta\delta^2 + \frac{eV_m}{2\pi h p_0}\cos\left(\frac{h\zeta}{R}\right) + \frac{eV_m \sin\Phi_s}{p_0 C}\zeta$$

Synchrotron tune

From linearization of the rf force around ζ=0, the synchrotron motion is reduced to a harmonic oscillator with a characteristic oscillation frequency (or tune, if divided by the beam mean revolution frequency)

$$\ddot{\zeta} + \underbrace{\left(\frac{eV_m h\eta\beta c}{p_0 CR}\cos\Phi_s\right)}_{\omega_s^2}\zeta = 0$$

$$Q_s = \frac{\omega_s}{\omega_0} = \sqrt{\frac{V_m[\mathrm{MV}]h\eta\cos\Phi_s}{2\pi\beta^2 E_0[\mathrm{MeV}]}}$$

$$\text{Note that: } \eta\cos\Phi_s = |\eta\cos\Phi_s| \geq 0$$

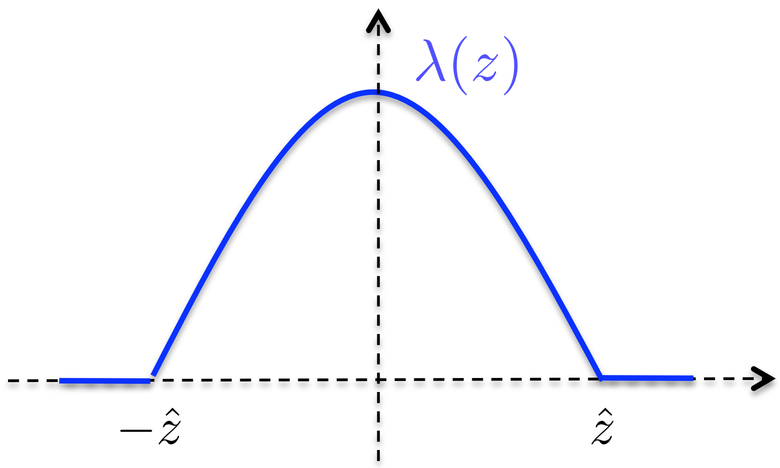
Synchrotron tune shift due to space charge

We consider the case of a parabolic bunch inside a stationary single rf bucket

$$\begin{cases} \dot{z} = -\eta\beta c\delta \\ \dot{\delta} = \dfrac{eV_m}{p_0 C}\sin\left(\dfrac{hz}{R}\right) - \dfrac{e^2 g\lambda'(z)}{2\pi\epsilon_0\gamma^2 p_0} \end{cases}$$

$$g = 0.33 + \log\frac{b}{a}$$

$$\lambda(z) = \begin{cases} \dfrac{3N_b}{4\hat{z}^3}(\hat{z}^2 - z^2) & \text{if } \ |z| \le \hat{z} \\ 0 & \text{if } \ |z| > \hat{z} \end{cases}$$



Synchrotron tune shift due to space charge

The rf force has to be linearized around z=0, while the space charge force is already linear with the chosen bunch line density.

$$\ddot{z} + \underbrace{\left(\frac{|\eta| e V_m h \beta c}{p_0 C R} + \frac{3 e^2 g N_b \eta \beta c}{4 \pi \epsilon_0 \gamma^2 \hat{z}^3 p_0} \right)}_{\omega_s^2 = \omega_0^2 Q_s^2 \approx \omega_0^2 \cdot (Q_{s0} + \Delta Q_s)^2} z = 0$$

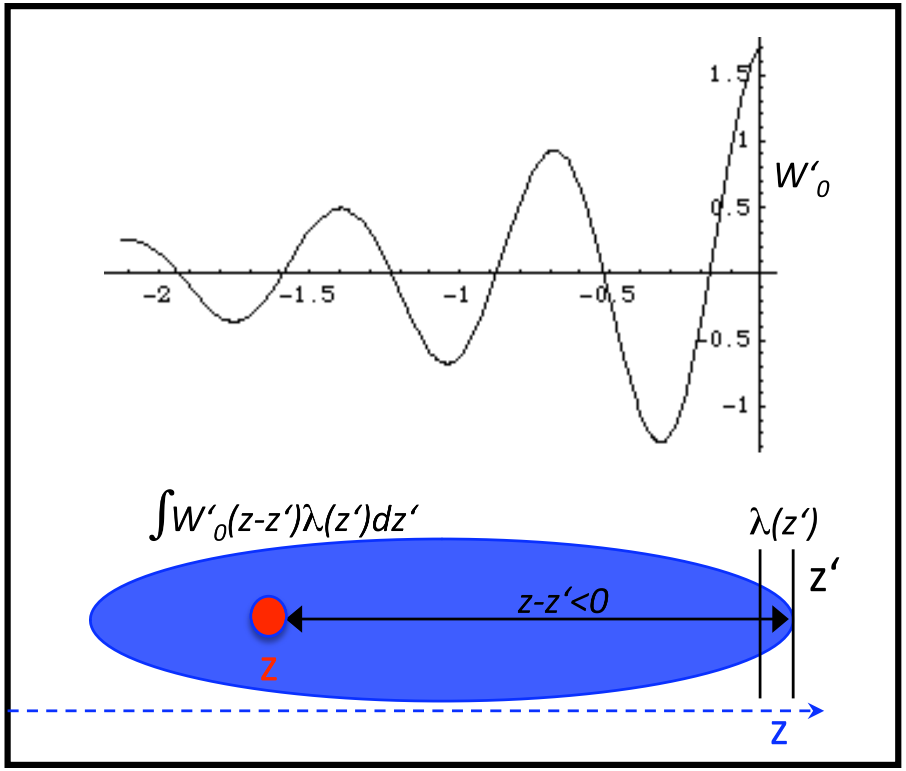
$$\Delta Q_s \approx \frac{3 e^2 g N_b \eta R^2}{8 \pi \epsilon_0 \beta^2 \gamma^2 \hat{z}^3 E_0 Q_{s0}}$$

Synchrotron tune shift due to space charge

- Remarkably, the space charge induced synchrotron tune shift depends on
  - Number of particles in the bunch N_b (linear)
  - Beam energy
    - Explicit inverse quadratic dependence on β and γ
    - There is another energy dependence in η. In particular the shift changes sign below and above transition. The tune shift is negative below transition ($\eta<0$), where space charge is defocusing. It is positive above transition ($\eta>0$), where space charge adds up to the external voltage and contributes with an extra focusing.
    - There is another inverse dependence on γ in E_0.
  - Machine radius (quadratic)
  - Bunch length (inverse cubic)
    - Attention must be paid here to the fact that the bunch length itself depends upon the strength of space charge….

General equations of longitudinal motion

- First of all we would like to write the full equations of the longitudinal dynamics by singling out all the different contributions (external voltage, space charge and wake field)
- We consider for the moment only wake fields decaying over one turn, or in the distance between two subsequent bunches of a train



$$F_w(z) = -\frac{e^2}{C}\int_z^{\infty} \lambda(z')W_0'(z-z')dz'$$

$$\text{since} \quad W_0'(z) = 0 \quad \text{for} \quad z > 0$$

$$F_w(z) = -\frac{e^2}{C}\int_{-\infty}^{\infty} \lambda(z')W_0'(z-z')dz' =$$

$$= -\frac{e^2}{C}(W_0' * \lambda)(z)$$

General equations of longitudinal motion

- Let's limit ourselves for now to the case of a stationary bucket.
- However, it should be noted that in the case of accelerating bucket, both the space charge term and the wake field term will be included into the momentum equation in a similar fashion, simply exchanging z with ζ, because both space charge fields and wake fields move with the bunch!

$$
\begin{cases}
\dot{z} = -\eta\beta c\delta \\
\dot{\delta} = \underbrace{\dfrac{eV_{\rm rf}(z)}{Cp_0}}_{\text{External rf}} - \underbrace{\dfrac{e^2 g\lambda'(z)}{2\pi\epsilon_0\gamma^2 p_0}}_{\text{Space charge}} - \underbrace{\dfrac{e^2}{Cp_0}\displaystyle\int_{-\infty}^{\infty}\lambda(z')W_0'(z-z')dz'}_{\text{Wake fields}}
\end{cases}
$$

Ex. Write the general Hamiltonian of this system, and specify it to the case of resistive wall and two in-phase rf systems on h=1 and h=3

Space charge impedance

- Now we will justify that we can always drop the space charge term from the general equations, because it can be included in the wake field formalism through an ad-hoc defined “space charge impedance”
- Therefore, we will be able to retain the only wake field term in the equations of the longitudinal dynamics.

$$W_0^{sc}(z) = \frac{gR}{\epsilon_0 \gamma^2}\delta(z) \Rightarrow [(W_0^{sc})' * \lambda](z) = \frac{g\lambda'(z)C}{2\pi\epsilon_0\gamma^2}$$

because $$\int_{-\infty}^{\infty} \lambda(z')W_0'(z-z')dz' = \int_{-\infty}^{\infty} \lambda'(z')W_0(z-z')dz'$$

$$Z_0^{sc}(\omega) = -\frac{i\omega}{c}F[W_0^{sc}(z)] = -\frac{i\omega}{c^2}\frac{gR}{\epsilon_0\gamma^2}$$

Space charge impedance

$$\frac{Z_0^{sc}(\omega)}{\omega} = -\frac{igR}{\epsilon_0 c^2 \gamma^2} \quad \Rightarrow \quad \frac{Z_0^{sc}(n)}{n} = -\frac{igR\omega_0}{\epsilon_0 c^2 \gamma^2}$$

$$n = \frac{\omega}{\omega_0}$$

- Longitudinal space charge can be easily included in the frame of the impedances through a newly defined "space charge impedance"
- The space charge impedance is purely reactive and exhibits a linear dependence on frequency. This is an approximation valid only in a range of frequencies well below the cut-off frequency of the beam pipe
- It is intuitive that the space charge impedance is proportional to the length of the accelerator and inversely proportional to the square of the beam energy.
- It will be shown in the following that, as purely imaginary, the space charge impedance does not contribute to the bunch energy loss. This is physically consistent with space charge being an internal force, which cannot determine a net energy loss over the full bunch (it can only happen that the energy lost through space charge by a part of the bunch, is transferred to another part of it). However, unlike in the transverse plane, here it can drive an instability (negative mass)

Energy loss of a bunch

$$\begin{cases} \dot{z} = -\eta\beta c\delta \\ \dot{\delta} = \dfrac{eV_{\mathrm{rf}}(z)}{Cp_0} - \dfrac{e^2}{Cp_0}\displaystyle\int_{-\infty}^{\infty} \lambda(z')W_0'(z-z')dz' \end{cases}$$

- We can write the equations in the above form without losing generality
- We are ready to calculate the energy loss due to the wakes
  - First the energy loss suffered by a particle at the z coordinate inside a bunch
  - Second, the energy loss suffered by the whole bunch

$$\Delta E(z) = -e^2 \int_{-\infty}^{\infty} \lambda(z')W_0'(z-z')dz'$$

$$\Delta E = \int_{-\infty}^{\infty} \Delta E(z)\lambda(z)dz = -e^2 \int_{-\infty}^{\infty} \lambda(z)dz \int_{-\infty}^{\infty} \lambda(z')W_0'(z-z')dz'$$

Energy loss of a bunch

- Even if single particles within the bunch can have a net energy loss or gain (sign of $\Delta E(z)$ not defined), as they can acquire more energy left behind by the preceding part of the bunch than what they lose on creating their own wake, the whole bunch can obviously only lose energy interacting with an external passive structure ($\Delta E<0$)
- The value $W'_0(0)$ accounts for the energy loss of the source of the wake
  - It can be seen by calculating the energy loss of a very short bunch (see below)
  - However, the final expression of the energy loss will be more complex because, as said above, part of the energy lost by a part of the bunch can be recovered by a following part of the bunch

$$W'_0(z-z') \rightarrow W'_0(0)$$

$$\Rightarrow \Delta E \approx -e^2 W'_0(0) \left[\int_{\Re} \lambda(z) dz \right]^2 = -e^2 W'_0(0) N_b^2$$

- $W'_0(0)$ quantifies the energy lost by the almost point-like (in z) source $q=N_b e$ on creating its own wake
- We assumed $W'_0(z)$ to be a continuous function in z, and thus also in z=0. This is true only in the non-ultrarelativistic case. See next page for the ultrarelativistic case.

Energy loss of a bunch

- In the ultrarelativistic case, since there cannot be any wake field in front of the bunch itself, when the bunch becomes very short the wake field tends to its limit for 0^-

$$W_0'(z-z') \to W_0'(0^-) \quad \Rightarrow \Delta E \approx -e^2 W_0'(0^-) \int_{\Re} \lambda(z)dz \int_z^{\infty} \lambda(z')dz' =$$

$$= -e^2 W_0'(0^-) \left[- \int_{N_b}^{0} u du \right] = -\frac{e^2 N_b^2}{2} W_0'(0^-)$$

- This result is the ***beam-loading theorem***
- The factor ½ comes from the fact that, because of the ultrarelativistic assumption, even an ultra-short bunch will only see in the average half of its charge.
- To keep the meaning of $W_0'(0)$ unchanged, we can define:

$$W_0'(z) = \begin{cases} W_0'(z) & \text{if} \quad z < 0 \\ \frac{1}{2} W_0'(0^-) & \text{if} \quad z = 0 \\ 0 & \text{if} \quad z > 0 \end{cases}$$

Energy loss of a bunch

- Now we go back to the energy loss of a bunch and try to express it as a function of the longitudinal impedance:

$$\Delta E = -e^2 \int_{\Re} \lambda(z)(\lambda * W_0')(z)dz = -\frac{e^2}{2\pi}\int_{\Re} \tilde{\lambda}^*(\omega)\tilde{\lambda}(\omega)Z_0^{||}(\omega)d\omega$$

$$\Delta E = -\frac{e^2}{2\pi}\int_{\Re} |\tilde{\lambda}(\omega)|^2 \mathrm{Re}[Z_0^{||}(\omega)]d\omega$$

- The energy loss only depends on the overlap between the real part of the longitudinal impedance and the bunch spectrum. Therefore, space charge (as well as any other purely reactive impedance) does not cause energy loss
- This result is only valid for the case of wake field decaying over one turn (broad-band impedance)

Energy loss of a bunch

- **Exercises:** calculate for a Gaussian bunch
  1. The energy loss per unit length due to a resistive pipe of radius b and conductivity σ
  2. The energy loss due to a broad band resonator (when needed make approximations on bunch length and resonator width)

$$\frac{Z_{rw}^{||}(\omega)}{L} = \sqrt{\frac{2}{\sigma Z_0 c}}\frac{Z_0}{4\pi b}|\omega|^{\frac{1}{2}}[1 - i\,\mathrm{sgn}(\omega)] \Rightarrow \frac{\Delta E}{L} = -\frac{N_b^2 e^2 c}{4\pi^2 b \sigma_z^{3/2}}\sqrt{\frac{Z_0}{2\sigma}} \cdot \Gamma\left(\frac{3}{4}\right)$$

$$Z_{BB}^{||}(\omega) = \frac{R_s}{1 + iQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)} \Rightarrow \begin{cases} \Delta E = -\frac{R_s e^2 N_b^2}{2\sqrt{\pi}Q^2\omega_r^2}\left(\frac{\beta^3 c^3}{\sigma_z^3}\right) & \text{if } \sigma_z \gg \frac{\beta c}{\omega_r} \\ \Delta E = -\frac{R_s e^2 N_b^2 \omega_r}{2Q} & \text{if } \sigma_z \ll \frac{\beta c}{\omega_r} \end{cases}$$

Energy loss of a bunch

- If the wake field decays very slowly and still has a significant amplitude by the time the bunch takes to go around the machine, bunch and wake field will interact on several passages
- In this case we take this into account into the calculation (assuming that the bunch distribution is basically stationary, i.e. does not significantly change from turn to turn -> $\lambda(z)$ periodic with period C).

$$\Delta E = -e^2 \int_{\Re} \lambda(z) dz \int_{\Re} dz' \lambda(z') \sum_{k=-\infty}^{\infty} W_0'(kC + z - z') dz'$$

We need to make use of the following general property of Fourier transforms

$$F(z) \longleftrightarrow \tilde{F}\left(\frac{\omega}{c}\right) \Longrightarrow \qquad \sum_{k=-\infty}^{\infty} F(kC) = \frac{c}{C} \sum_{p=-\infty}^{\infty} \tilde{F}\left(\frac{2\pi p c}{C}\right)$$

Energy loss of a bunch

- From the Eq. on the previous page, we derive a closed expression for the multi-turn energy loss of a bunch (can be also multi bunch, if we can approximate the finite summation on the number of bunches, M, with an infinite summation

$$\sum_{k=-\infty}^{\infty} W_0'(kC + z - z') = \frac{c}{C} \sum_{p=-\infty}^{\infty} Z_0^{||}(p\omega_0) \exp\left[-\frac{ip\omega_0(z-z')}{c}\right]$$

$$\Delta E = -\frac{e^2\omega_0}{2\pi} \sum_{p=-\infty}^{\infty} Z_0^{||}(p\omega_0) \underbrace{\int_{\Re} \lambda(z) \exp\left(\frac{-ip\omega_0 z}{c}\right) dz}_{\tilde{\lambda}(p\omega_0)} \underbrace{\int_{\Re} \lambda(z') \exp\left(\frac{ip\omega_0 z'}{c}\right) dz'}_{\tilde{\lambda}^*(p\omega_0)}$$

$$\Delta E = -\frac{e^2\omega_0}{2\pi} \sum_{p=-\infty}^{\infty} |\tilde{\lambda}(p\omega_0)|^2 \mathrm{Re}[Z_0^{||}(p\omega_0)]$$

Energy loss of a bunch

- **Exercises:**
  1. Discuss when the multi-turn formula for the energy loss reduces to that for single pass
  2. Using the single pass and multi-turn results, discuss the multi-bunch case for different filling patterns of the machine
     a) M bunches uniformly distributed (PSB, M=h=1 or M=h=2)
     b) M bunches separated by C/h, with h>>M (long gap) (SPS, M=72, h=4620)
     c) M bunches separated by C/h with h≈M (short gap) (PS, M=6, h=7)

$$\omega_0 \sum_{p=-\infty}^{\infty} \rightarrow \int_{-\infty}^{\infty} d\omega$$

$V_{rf}(z)$: use of second harmonic

$$\begin{cases} \dot{z} = -\eta \beta c \delta \\ \\ \dot{\delta} = \dfrac{eV_{\rm rf}(z)}{Cp_0} - \dfrac{e^2}{Cp_0} \displaystyle\int_{-\infty}^{\infty} \lambda(z') W_0'(z - z') dz' \end{cases}$$

- General equations of longitudinal motion for a bunch in a stationary bucket
- $V_{rf}(z)$ depends on the specific rf system and can contain several harmonics, determining eventually, together with the wake fields, the shape of the bunch
- An important example is when $V_{rf}(z)$ has the contribution of only two harmonics, h_1 and h_2=2h_1. Assuming the lower harmonic as the main harmonic, the second harmonic can lengthen or shorten the bunch (BL or BS mode) according to its relative phase to the first harmonic

$V_{rf}(z)$: use of second harmonic

- In BS mode focusing must be strengthened, so the two harmonics need to be in phase
- In BL mode, focusing is weakened and the two harmonics are out of phase
- Flattening of the bunch in the center can be optimized by making the slopes of the two forces around the origin equal and opposite in sign (equivalently, requiring first and second derivatives of the force to be 0 in the origin)

$$\begin{aligned} &\text{BL mode} \longrightarrow \Phi_{12} = \pi \\ &\text{BS mode} \longrightarrow \Phi_{12} = 0 \end{aligned}$$

$$V_{\mathrm{rf}}(z) = \mathrm{sgn}(\eta) \cdot \begin{cases} V_{m1} \sin\left(\frac{h_1 z}{R}\right) + V_{m2} \sin\left(\frac{2h_1 z}{R}\right) & \text{BS} \\ V_{m1} \sin\left(\frac{h_1 z}{R}\right) - V_{m2} \sin\left(\frac{2h_1 z}{R}\right) & \text{BL} \end{cases}$$

$$V_{m1}\frac{h_1}{R}z - V_{m2}\frac{2h_1}{R}z = 0 \quad \Longrightarrow \quad \frac{V_{m2}}{V_{m1}} = 0.5$$

Hamiltonian of the system

- We write the general Hamiltonian of the system, using a general potential for the rf force, U(z), which can be specified for each case

$$H = -\frac{1}{2}\eta\beta c\delta^2 + \frac{e}{Cp_0}U(z) + \frac{e^2}{Cp_0}\int_{-\infty}^{z} dz'' \int_{\Re} \lambda(z')W_0'(z'' - z')dz'$$

Examples

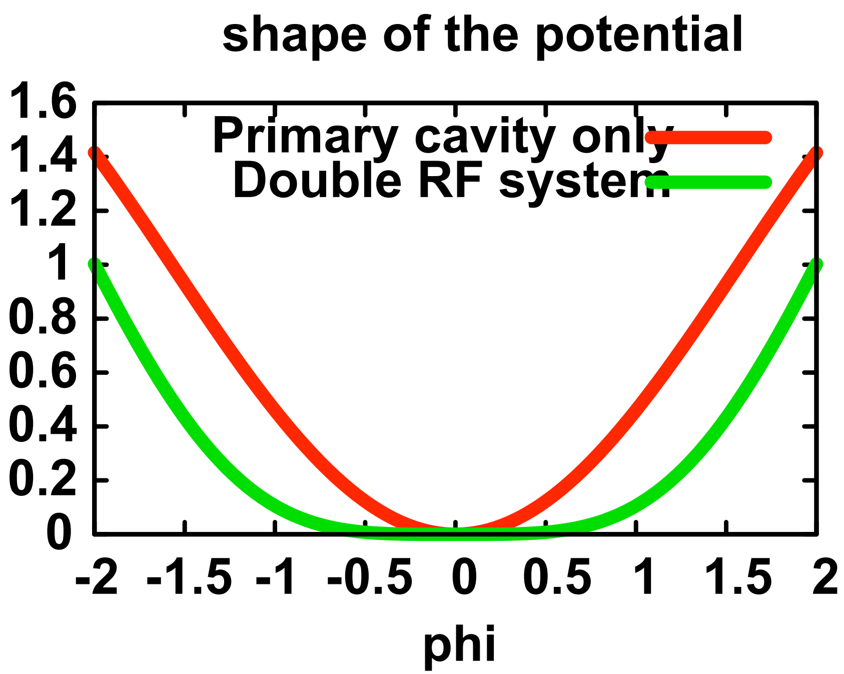
$$U(z) = \frac{V_m R}{h}\cos\left(\frac{hz}{R}\right)\cdot \mathrm{sgn}(\eta)$$

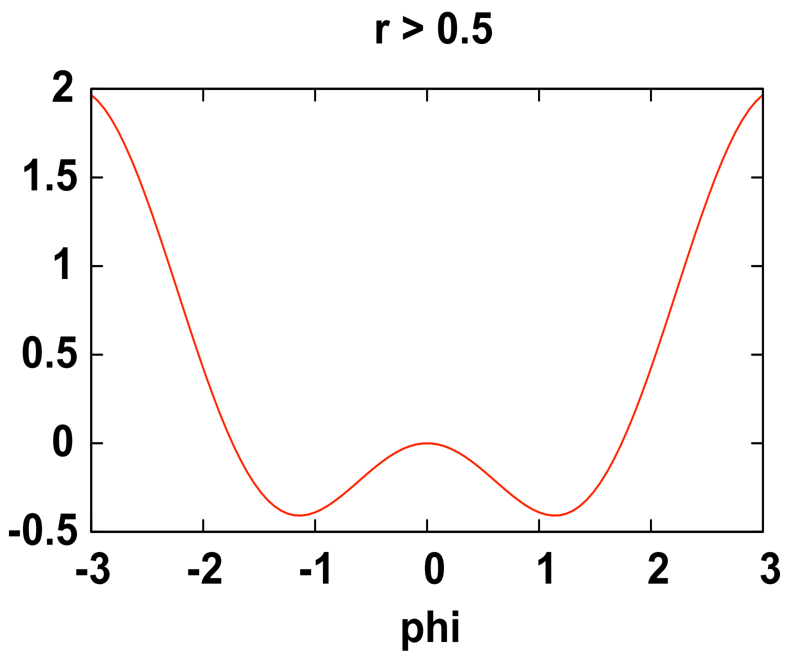
$$U(z) = \frac{V_{m1} R}{h_1}\left[\cos\left(\frac{h_1 z}{R}\right) \pm \frac{1}{4}\cos\left(\frac{2h_1 z}{R}\right)\right]\cdot \mathrm{sgn}(\eta)$$

$$U(z) = -\frac{V_m h}{2R}z^2 \cdot \mathrm{sgn}(\eta)$$

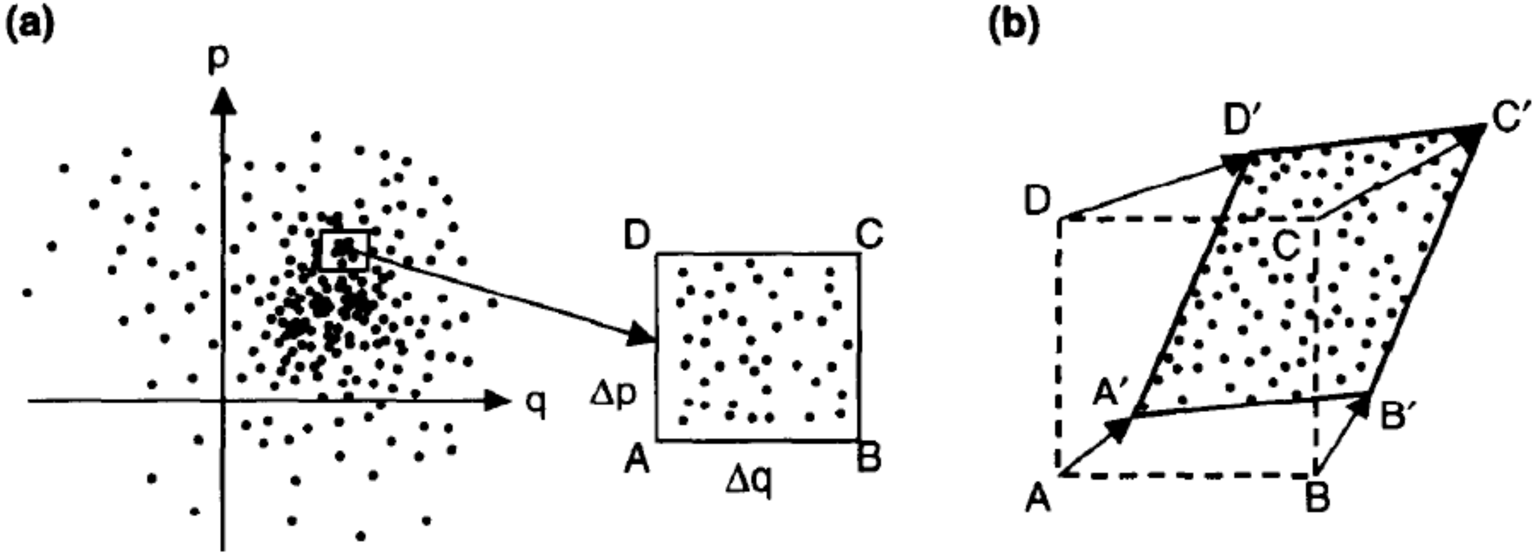
Hamiltonian of the system

- Examples of U(z) for the case of single harmonic and double harmonic with different voltage ratios (r=0.5 and a case r>0.5)
- The case r=0.5 shows perfect flattening, but some times a ratio r>0.5 is used in order to increase the bunching factor further and reduce the effect of transverse space charge.



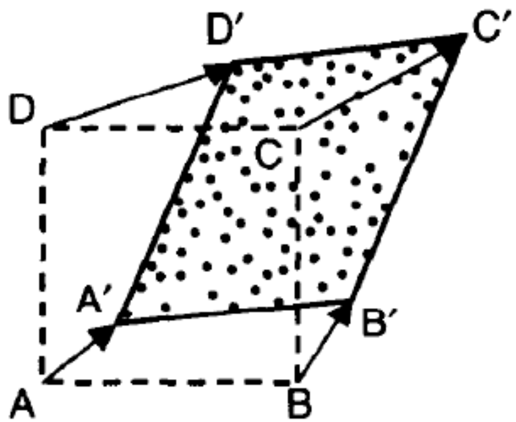


Vlasov equation



- Let's consider a phase space distribution of particles such that
  - There are no mechanisms of generation/loss
  - Single particles evolve following a Hamiltonian law of motion
- This distribution behaves like an uncompressible fluid, i.e. has the property that whatever area occupied by a fraction of the particles is conserved during the evolution (Hamiltonian flow)
- This entails that the total time derivative of the distribution is zero

Vlasov equation



$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} \\ \dot{p} = -\dfrac{\partial H}{\partial q} \end{cases}$$

$$\psi(q,p,t)\text{area}(ABCD) = \psi(q+\dot{q}dt, p+\dot{p}dt, t+dt)\text{area}(A'B'C'D')$$

$$\psi(q,p,t) = \psi(q+\dot{q}dt, p+\dot{p}dt, t+dt) \quad \Rightarrow \quad \frac{d\psi}{dt} = 0$$

$$\frac{\partial \psi}{\partial t} + \dot{q}\frac{\partial \psi}{\partial q} + \dot{p}\frac{\partial \psi}{\partial p} = 0$$

Vlasov equation

$$\begin{cases} \dot{z} = -\eta\beta c\delta \\ \dot{\delta} = \dfrac{eV_{\rm rf}(z)}{Cp_0} - \dfrac{e^2}{Cp_0}\displaystyle\int_{-\infty}^{\infty} \lambda(z')W_0'(z-z')dz' \end{cases}$$

- The beam (bunch) is described by a distribution function *$\Psi(z,\delta,t)$*, solution of the Vlasov equation
- The stationary (equilibrium) solutions *$\Psi_0(z,\delta)$* can be easily found because they must be functions of the Hamiltonian of the system

$$\psi_0(z,\delta) = \psi_0(H) \quad \Rightarrow \quad \begin{aligned} &\dot{z}\frac{\partial\psi_0}{\partial H}\frac{\partial H}{\partial z} + \dot{\delta}\frac{\partial\psi_0}{\partial H}\frac{\partial H}{\partial \delta} = \\ &= \frac{\partial H}{\partial \delta}\frac{\partial\psi_0}{\partial H}\frac{\partial H}{\partial z} - \frac{\partial H}{\partial z}\frac{\partial\psi_0}{\partial H}\frac{\partial H}{\partial \delta} = 0 \end{aligned}$$

Stationary solutions

General solution with wake fields

$$\psi_0\left[-\frac{1}{2}\eta\beta c\delta^2+\frac{eU(z)}{Cp_0}+\frac{e^2}{Cp_0}\int_{-\infty}^{z}dz"\int_{\Re}W_0'(z"-z')\lambda(z')dz'\right]$$

General solution without wake fields

$$\psi_0\left[-\frac{1}{2}\eta\beta c\delta^2+\frac{eU(z)}{Cp_0}\right]$$

Distribution has to satisfy the normalization condition

$$\iint_{\Re^2}\psi(z,\delta)dzd\delta=N_b$$

Stationary solutions, matching (no wake fields)

Let's use the linearized potential (bunch much shorter than the bucket)

$$U(z) = -\frac{V_m h}{2R} z^2 \cdot \mathrm{sgn}(\eta)$$

and we choose an exponential solution (double Gaussian)

$$\psi_0(z,\delta) = k_N \exp\left[-\frac{1}{H_0}\left(\frac{1}{2}|\eta|\beta c\delta^2 + \frac{V_m e h z^2}{2CRp_0}\right)\right]$$

This corresponds to a simply Gaussian momentum distribution

$$\psi_0(\delta) = \int_{\Re} \psi_0(z,\delta)dz \propto \exp\left[-\frac{1}{2H_0}|\eta|\beta c\delta^2\right]$$

$$\sigma_\delta^2 = \frac{H_0}{|\eta|\beta c} \quad \Rightarrow \quad H_0 = |\eta|\beta c\sigma_\delta^2$$

Stationary solutions, matching (no wake fields)

And also Gaussian distribution in z

$$\lambda(z) = \int_{\Re} \psi(z,\delta) d\delta \propto \exp\left(-\frac{V_m e h z^2}{2CRp_0H_0}\right)$$

$$\sigma_z^2 = \frac{CRp_0H_0}{ehV_m} = \frac{CRE_0\beta^2|\eta|\sigma_\delta^2}{ehV_m} = \frac{R^2\eta^2\sigma_\delta^2}{Q_s^2}$$

The extensions of line density and momentum distribution, quantified by their rms values, are not independent, but related through the **matching condition**

$$\frac{R|\eta|\sigma_\delta}{Q_s\sigma_z} = 1$$

Stationary solutions, matching (no wake fields)

A possible equilibrium solution for the short bunch is therefore

$$\psi_0(z,\delta) = \frac{N_b}{2\pi\sigma_z\sigma_\delta}\exp\left[-\left(\frac{z^2}{2\sigma_z^2}+\frac{\delta^2}{2\sigma_\delta^2}\right)\right]$$

- σ_z and σ_δ must be determined through
  - Lepton machine: Gaussian solution matches naturally the problem, because the Gaussian distribution in momentum, as well as its rms value σ_δ, are fixed by the equilibrium between radiation damping and quantum excitation. The σ_z is then given by the matching condition
  - Hadron machine: Gaussian distribution could be incorrect or inaccurate. However, we usually know the longitudinal emittance (conserved through a chain in absence of any, voluntary or not, blow up mechanism) and can therefore determine σ_z and σ_δ of a Gaussian solution from the emittance (proportional to the product $\gamma\sigma_z\sigma_\delta$) and the matching condition
- The matching condition is true only for short bunches, the problem of nonlinear matching will be considered later

Stationary solutions, matching (no wake fields)

For hadrons, some times an elliptical distribution in phase space could be preferable, which corresponds to parabolic distributions in z and δ (**exercise 1**)

$$\psi_0(H) = \begin{cases} k_N \sqrt{1 + \frac{H}{H_0}} & \text{if } \ |H| \leq |H_0| \\ 0 & \text{if } \ |H| > |H_0| \end{cases}$$

$$\psi_0(z, \delta) = \begin{cases} k_N \sqrt{1 - \left(\frac{z}{\hat{z}}\right)^2 - \left(\frac{\delta}{\hat{\delta}}\right)^2} & \text{if } \ \left(\frac{z}{\hat{z}}\right)^2 + \left(\frac{\delta}{\hat{\delta}}\right)^2 \leq 1 \\ 0 & \text{if } \ \left(\frac{z}{\hat{z}}\right)^2 + \left(\frac{\delta}{\hat{\delta}}\right)^2 > 1 \end{cases}$$

$$\frac{\hat{\delta}|\eta| R}{Q_s \hat{z}} = 1$$

Exercise 2: prove that the matching condition is satisfied also for an elliptical distribution

Stationary solutions, nonlinear matching (no wake fields)

When the bunch occupies a vast area in the longitudinal phase space (comparable to the bucket area), matching is more complex

The exponential solution has to be cut at the limits of the stability region in order to give a possible stationary solution of Vlasov equation

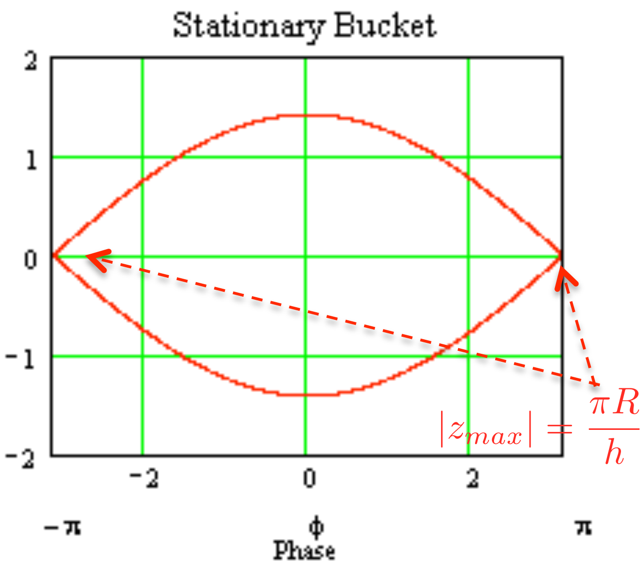
$$\psi_0(z,\delta) = k_N \exp\left(-\frac{\eta\beta c\delta^2}{2H_0}\right)\exp\left(\frac{eU(z)}{Cp_0H_0}\right)\Omega(H)$$

$$\Omega(H) = \begin{cases} 1 & \text{in domain of stability} \\ 0 & \text{outside} \end{cases}$$

The equation of the domain of stability is usually given by the equation $H=H_{max}$, with H_{max} being the Hamiltonian of the system at the first unstable fixed point(s). Typically for hadrons, because lepton machines are designed with very large buckets in order to improve the Touschek lifetime of the beam.

Stationary solutions, nonlinear matching (no wake fields)

Let's consider for instance the case of a long bunch inside a single harmonic stationary bucket (e.g. the LHC beam at the injection plateau in the SPS)



$$H = -\frac{1}{2}\eta\beta c\delta^2 - \text{sgn}(\eta)\frac{eV_m}{2\pi h p_0}\left[1 - \cos\left(\frac{hz}{R}\right)\right]$$

$$|H_{max}| = |H(z_{max}, 0)| = \frac{eV_m}{\pi h p_0}$$

$$\text{stability boundary} \;\rightarrow\; |H(z,\delta)| = H_{max}$$

$$\lambda(z) = \int_{\Re} \psi_0(z,\delta)d\delta = k_N \int_{-\sqrt{2}\frac{Q_s}{h|\eta|}\sqrt{1+\cos\left(\frac{hz}{R}\right)}}^{\sqrt{2}\frac{Q_s}{h|\eta|}\sqrt{1+\cos\left(\frac{hz}{R}\right)}} \exp\left(-\frac{1}{2H_0}|\eta|\beta c\delta^2\right)\exp\left[\frac{eV_m}{Cp_0H_0}\cos\left(\frac{hz}{R}\right)\right]d\delta$$

Stationary solutions, nonlinear matching (no wake fields)

If we know the bunch length, we can write the line density in closed form, determine the two constants k and H_0 by using the conditions below and then calculate the only possible rms momentum spread that solves the problem.
If we know the longitudinal emittance as an input, the nonlinear matching requires expressing the momentum distribution, as well, and the problem becomes more involved.

$$\lambda(z) = \tilde{k} \cdot \exp\left[\frac{eV_m}{Cp_0H_0}\cos\left(\frac{hz}{R}\right)\right] \operatorname{erf}\left[\frac{2Q_s}{h|\eta|}\sqrt{\frac{H_0}{|\eta|\beta c}\left[1+\cos\left(\frac{hz}{R}\right)\right]}\right]$$

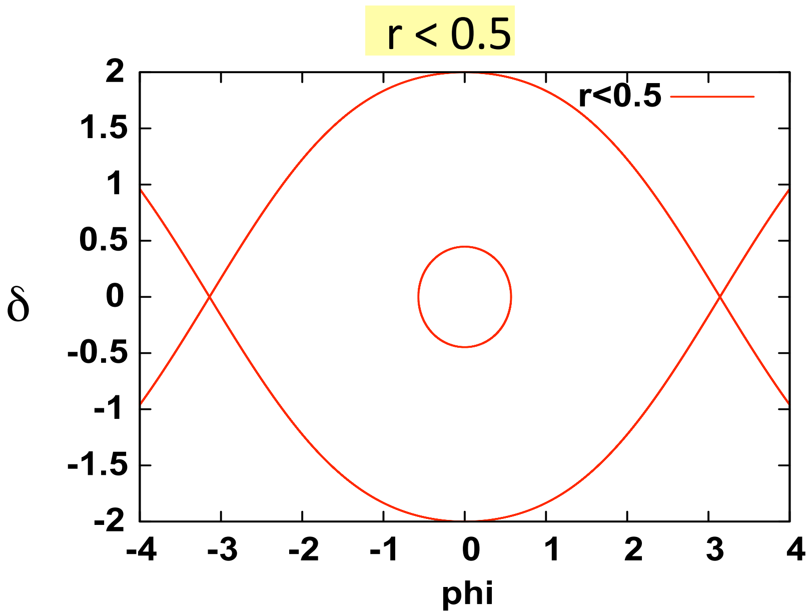
$$\int_{\Re} \lambda(z)dz = N_b \qquad \sigma_z^2 = \frac{1}{N_b}\int_{\Re} z^2\lambda(z)dz$$

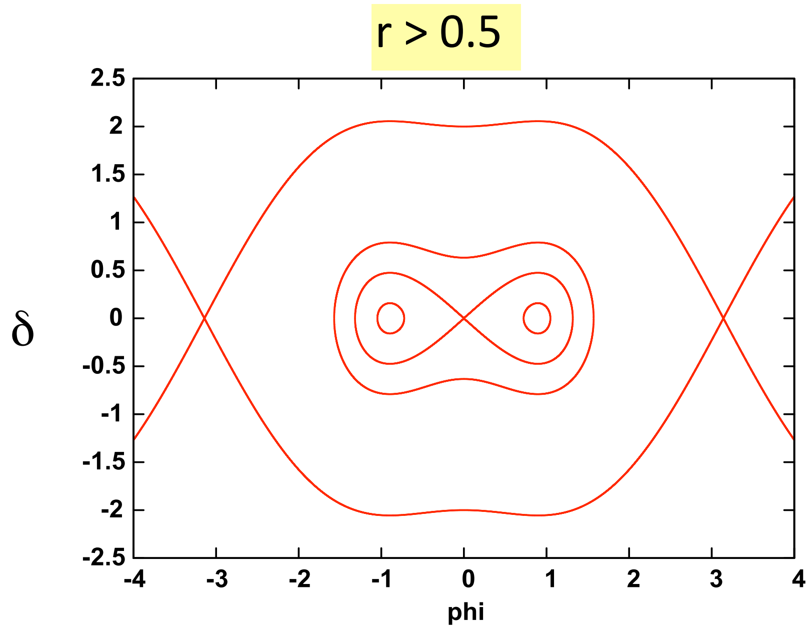
Stationary solutions, nonlinear matching (no wake fields)

In a double harmonic rf system the boundary of the stability is different

⇒ if the ratio between the voltage is below 0.5, the origin of the longitudinal phase space is still a stable point and the effect of the second harmonic is just to flatten the separatrices around it

⇒ If the ration is above 0.5, the origin becomes an unstable fixed point, but the trajectories around it are still limited and the total stable area increases further.





Stationary solutions, unmatched case (no wake fields)

But what happens when the beam parameters do not fulfill the matching condition (linear or nonlinear)? In other words, what happens if we inject a beam into a machine with length and momentum spread not matched with the waiting bucket?

⇒ No stationary solutions exist, the beam parameters at injection have to be used as initial condition to solve the Vlasov equation in time varying regime

⇒ The problem can also be solved using the envelope equation formalism (see yesterday's lecture from Elias).

⇒ In the linear case: the beam will start rotating in phase space, at a frequency that is twice the single particle synchrotron tune. This is called a "quadrupole oscillation" and is undamped, i.e. it would in principle last forever.

⇒ In the nonlinear case, the quadrupole oscillation dies out in time at the expense of longitudinal emittance increase (dilution) caused by phase space filamentation. Since particles at large amplitudes rotate at slower frequencies, they will be left behind while the core is executing the oscillation at twice the nominal synchrotron frequency. The beam will decohere and eventually occupy a larger region, matching itself naturally to the bucket. In this case, there exists an asymptotic solution, but again, not a stationary solution.

⇒ Mismatching can be done on purpose, e.g. to have a fast bunch rotation or longitudinal emittance blow-up

Stationary solutions with wake fields
Potential well distortion

We study now the equilibrium bunch distribution in presence of wake fields.
We take the linear approximation for the rf force (quadratic potential)

$$\psi_0(z,\delta) = k_N \exp\left[-\frac{1}{H_0}\left(\frac{1}{2}|\eta|\beta c\delta^2 + \frac{Q_s^2\beta c}{|\eta|R^2}z^2 - \frac{e^2\mathrm{sgn}(\eta)}{Cp_0}\int_{-\infty}^{z}dz" \int_{\Re} W_0'(z"-z')\lambda(z')dz'\right)\right]$$

From the momentum distribution: $H_0 = \sigma_\delta^2|\eta|\beta c$

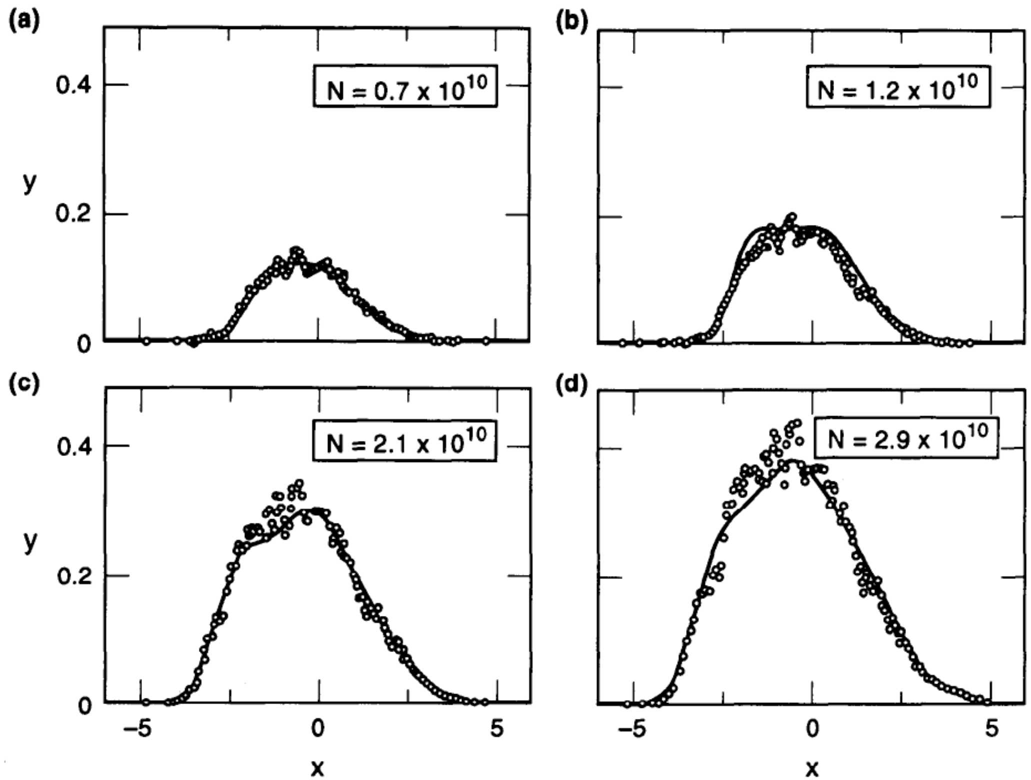
$$\lambda(z) = k_N\sqrt{2\pi}\sigma_\delta \exp\left[-\frac{Q_{s0}^2 z^2}{\eta^2\sigma_\delta^2 R^2} + \frac{e^2}{CE_0\beta^2\eta\sigma_\delta^2}\int_{-\infty}^{z}dz" \int_{\Re} W_0'(z"-z')\lambda(z')dz'\right]$$

Haissinski equation

Stationary solutions with wake fields Potential well distortion

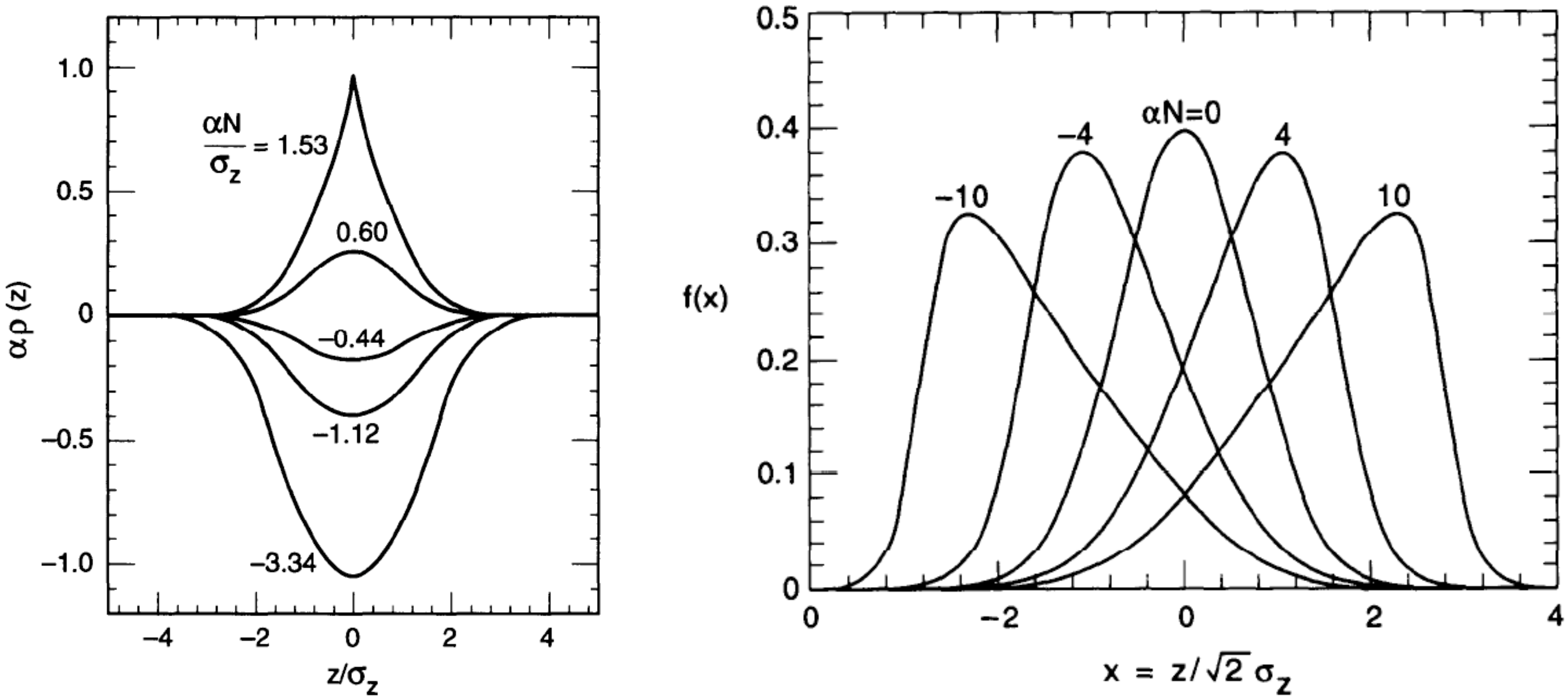
- The Haissinski equation is an implicit integral equation in the unknown $\lambda(z)$.
  - Lepton machines: The momentum spread σ_δ is given, and we can use the equation to determine the line density alone, with the coefficient in front solely defined by the normalization condition. Since the solution for the momentum distribution is Gaussian, this type of solution is specially suited for this type of machines.
  - Hadron machines: The longitudinal emittance is given. In this case a joint solution ($\lambda(z)$, σ_δ) must be sought such that the resulting pair (σ_z, σ_δ) satisfies the condition on the longitudinal emittance.
- A way of solving Haissinski equation numerically consists in using successive iterations. A first order perturbed $\lambda_1(z)$ is calculated by plugging the unperturbed Gaussian shape in the RHS of the equation, then $\lambda_2(z)$ will be calculated by plugging $\lambda_1(z)$ in the RHS, and so on until the procedure converges.
- An analytical treatment of the problem can be made assuming that the wake fields only produce a small deviation from the unperturbed solution
  - The bunch shape stays Gaussian
  - The center of the bunch moves from $z=0$ to $z=z_0$, such that the associated synchronous phase $\Phi_s=(hz_0)/R$ ensures constant compensation for the energy loss caused by the real part of the impedance
  - The bunch length will change, such as to re-match the bunch to the distorted bucket. This adjustment can produce shortening or lengthening with respect to the unperturbed solution, according to whether the impedance produces more or less net focusing.

Stationary solutions with wake fields
Potential well distortion



- Numerical solution of the Haissinski equation for the electron damping ring for the SLAC linear collider
- Bunch shape is Gaussian at low intensity
- Distortion occurs as intensity increases. In particular, the distribution tends to lean forward (z>0) in order to compensate with the rf for the parasitic loss due to the impedance

Stationary solutions with wake fields
Potential well distortion



- Solution of the Haissinski equation for the two special cases:
  - Reactive impedance (left plot), iSZ_0. The parameter α is proportional to S/η, so that $\alpha>0$ means either above transition+capacitive impedance or below transition+inductive impedance. In both cases the bunch shortens.
  - Resistive impedance (right plot), SZ_0. Below transition ($\alpha<0$) the bunch moves backward, and it moves forward above transition. In both cases the bunch takes power from the rf to compensate for the energy loss.

Stationary solutions with wake fields
Potential well distortion

- For small deviation, we assume then that the bunch distribution stays Gaussian (ansatz)

$$\lambda(z) \propto \exp\left[-\frac{(z-z_0)^2}{2\sigma_z^2}\right]$$

- We can also expand in Taylor series around z=0 the wake field contribution to the Hamiltonian as it appears in the Haissinski equation.

$$\lambda(z) \propto \exp\left[-\frac{z^2}{2\sigma_{z0}^2} - (k_1 z + k_2 z^2)\right]$$

$$\begin{cases} \sigma_z = \dfrac{\sigma_{z0}}{\sqrt{1+2\sigma_{z0}^2 k_2}} \\ z_0 = -k_1 \sigma_z^2 \end{cases}$$

N.B. Rigorously speaking, these are still implicit equations, because $k_1=k_1(\sigma_z)$ and $k_2=k_2(\sigma_z)$, but they are not functions of z_0 because the wake moves with the bunch

Stationary solutions with wake fields
Potential well distortion

- Calculation of k_2 first
- This is the second order term in the Taylor expansion of the wake field contribution to the Hamiltonian

$$k_2 = -\frac{e^2}{C E_0 \beta^2 \eta \sigma_\delta^2} \cdot \frac{1}{2} \underbrace{\left\{ \frac{d^2}{dz^2} \left[\int_{-\infty}^{z} dz'' \int_{\Re} W_0'(z''-z')\lambda(z')dz' \right] \right\}_{z=0}}_{\int_{\Re} W_0''(-z')\lambda(z')dz'}$$

$$\int_{\Re} W_0''(-z')\lambda(z')dz' = -\int_{\Re} \tilde{\lambda}(\omega)\frac{i\omega}{c} Z_0^{||}(\omega)\frac{d\omega}{2\pi} = \frac{1}{2\pi c}\int_{\Re} \tilde{\lambda}(\omega)\omega \mathrm{Im}[Z_0^{||}(\omega)]d\omega$$

$$k_2 = -\frac{e^2}{C E_0 \beta^2 \eta \sigma_\delta^2} \cdot \frac{1}{2\pi c}\int_{\Re} \tilde{\lambda}(\omega)\omega \mathrm{Im}[Z_0^{||}(\omega)]d\omega$$

Stationary solutions with wake fields
Potential well distortion

- The sign of k_2 determines whether there is bunch lengthening or bunch shortening ($K_2<0 \Rightarrow \sigma_z>\sigma_{z0}$ and vice versa).
- From the expression on the previous page, the sign of k_2 is jointly determined by the sign of η and that of the integral, which in turn depends on the overlap between the impedance and the bunch spectrum.
- For Gaussian bunch and space charge-type impedance $-iSZ_0\omega$:

$$\int_{\Re} \tilde{\lambda}(\omega)\omega \mathrm{Im}[Z_0^{||}(\omega)]d\omega = -SZ_0 \int_{\Re} \omega^2 \tilde{\lambda}(\omega)d\omega =$$

$$= -\sqrt{2\pi} SZ_0 \sigma_\omega^3 N_b = -\frac{\sqrt{2\pi} SZ_0 c^3 N_b}{\sigma_{z0}^3}$$

$$k_2 \begin{cases} < 0 & \text{below transition} \\ > 0 & \text{above transition} \end{cases}$$

Stationary solutions with wake fields
Potential well distortion

- Calculation of k_1
- This is the first order term in the Taylor expansion of the wake field contribution to the Hamiltonian

$$- k_1 = \frac{e^2}{CE_0\beta^2\eta\sigma_\delta^2} \underbrace{\left\{ \frac{d}{dz} \left[\int_{-\infty}^{z} dz'' \int_{\Re} W_0'(z''-z')\lambda(z')dz' \right] \right\}_{z=0}}_{\int_{\Re} W_0'(-z')\lambda(z')dz'}$$

$$\int_{\Re} W_0'(-z')\lambda(z')dz' = \int_{\Re} \tilde{\lambda}(\omega) Z_0^{||}(\omega) \frac{d\omega}{2\pi} = \frac{1}{2\pi} \int_{\Re} \tilde{\lambda}(\omega) \mathrm{Re}[Z_0^{||}(\omega)] d\omega$$

$$- k_1 = \frac{e^2}{CE_0\beta^2\eta\sigma_\delta^2} \cdot \frac{1}{2\pi} \int_{\Re} \tilde{\lambda}(\omega) \mathrm{Re}[Z_0^{||}(\omega)] d\omega$$

The bunch will move to the left ($z_0<0$) below transition and to the right ($z_0>0$) above transition

Stationary solutions with wake fields
Synchrotron tune shift

$$H = -\frac{1}{2}|\eta|\beta c\delta^2 - \frac{\beta c Q_s^2}{|\eta| R^2} z^2 + \mathrm{sgn}(\eta)\frac{e^2}{C p_0}\int_{-\infty}^{z} dz'' \int_{\Re} W_0'(z''-z')\lambda(z')dz'$$

We still rely on the second order term in z in the Taylor expansion of the wake field contribution to the Hamiltonian

$$-\frac{\beta c}{|\eta| R^2} z^2 \left[Q_s^2 - \frac{e^2 \eta R^2}{C p_0 \beta c}\cdot\frac{1}{4\pi c}\int_{\Re}\tilde{\lambda}(\omega)\omega \mathrm{Im}[Z_0^{||}(\omega)]d\omega\right]$$

$$\Delta Q_s \simeq -\frac{1}{4}\frac{e^2\eta}{(2\pi)^2 p_0 \omega_s}\int_{\Re}\tilde{\lambda}(\omega)\omega \mathrm{Im}[Z_0^{||}(\omega)]d\omega$$

Exercise: how should the synchrotron tune move due to space charge ? Verify that with the above formula it goes in the right direction and the formula on slide 22 is recovered

Stationary solutions with wake fields
Generalization to multi-pass

All the formulae obtained so far to describe the potential well distortion can be generalized in the multi-turn regime in a similar fashion as was done for the energy loss!

Rule: $$\int_{\Re} f(\omega)d\omega \;\rightarrow\; \omega_0 \sum_{p=-\infty}^{\infty} f(p\omega_0)$$

$$k_2 = -\frac{e^2}{C^2 E_0 \beta\eta\sigma_\delta^2} \sum_{p=-\infty}^{\infty} \tilde{\lambda}(p\omega_0) p\omega_0 \mathrm{Im}[Z_0^{||}(p\omega_0)]$$

$$k_1 = -\frac{e^2}{C^2 p_0 \eta\sigma_\delta^2} \sum_{p=-\infty}^{\infty} \tilde{\lambda}(p\omega_0) \mathrm{Re}[Z_0^{||}(p\omega_0)]$$

$$\Delta Q_s = -\frac{e^2\eta}{4(2\pi)^2 p_0 Q_s} \sum_{p=-\infty}^{\infty} \tilde{\lambda}(p\omega_0) p\omega_0 \mathrm{Im}[Z_0^{||}(p\omega_0)]$$

Stationary solutions with wake fields
Potential well distortion

- **Exercise**: calculate center shift and bunch length of a bunch under the action of:

1. A purely resistive impedance

$$Z_0^{||}(\omega) = \frac{SZ_0}{4\pi}$$

2. A purely reactive impedance

$$Z_0^{||}(\omega) = i\frac{SZ_0}{4\pi}\frac{\omega}{\omega_0}$$

Another possibility to calculate the center shift from the Haissinski equation imposing the condition:

$$\lambda'(z_0) = 0$$

Linearization of Vlasov equation

- First, we need to change the system of coordinates
- We define polar coordinates (r, Φ), such that the unperturbed Hamiltonian can be written as a function of r alone, which translates into unperturbed solutions of Vlasov equation of the type $\Psi_0(r)$.

$$z = r\cos\Phi$$

$$\frac{\eta\beta c}{\omega_s}\delta = r\sin\Phi$$

$$H = \frac{1}{2}\omega_s r^2$$

- We will search for wave-like solutions of the Vlasov equation in the form below
- Since $\Psi_0(r)$ is the solution of the unperturbed Vlasov equation, the potential well distortion will be also included in the wave term (Ω=0)

$$\psi(r,\Phi,t) = \psi_0(r) + \psi_1(r,\Phi)\exp\left(-i\Omega t\right)$$

Linearization of Vlasov equation

- We also write the wake field term in the momentum equation directly in a form that includes the multi-turn contribution to the force

$$F_w(z,t) = -\frac{e^2}{C} \sum_{-\infty}^{\infty} \int_{\Re} \lambda(z', t - kT_0) W_0'(z - z' - kC) dz'$$

$$\lambda(z,t) = \lambda_0(z) + \lambda_1(z) \exp(-i\Omega t)$$

$$F_w(z,t) = -\frac{e^2}{C} \int_{\Re} \lambda_1(z') dz' \sum_{-\infty}^{\infty} \exp[-i\Omega(t - kT_0)] W_0'(z - z' - kC)$$

$$F_w(z,t) = -\frac{e^2}{C} \exp(-i\Omega t) \sum_{k=-\infty}^{\infty} \exp\left(i\Omega \frac{kC}{c}\right) \int_{\Re} \lambda_1(z') W_0'(z - z' - kC) dz'$$

Linearization of Vlasov equation

- We recast the wake field term in the momentum equation in a form that contains an explicit dependence on the longitudinal impedance

$$\sum_{k=-\infty}^{\infty} \left[\exp\left(\frac{i\Omega\tilde{z}}{c}\right) (\lambda_1 * W_0')(z-\tilde{z}) \right]_{\tilde{z}=kC}$$

$$\Updownarrow$$

$$\frac{1}{T_0} \sum_{p=-\infty}^{\infty} \exp\left[i(p\omega_0+\Omega)\frac{z}{c} \right] \tilde{\lambda}_1(p\omega_0+\Omega) Z_0^{||}(p\omega_0+\Omega)$$

$$F_w(z,t) = -\frac{e^2}{CT_0}\exp(-i\Omega t) \sum_{p=-\infty}^{\infty} \exp\left[i(p\omega_0+\Omega)\frac{z}{c} \right] \tilde{\lambda}_1(p\omega_0+\Omega) Z_0^{||}(p\omega_0+\Omega)$$

Linearization of Vlasov equation

- Next step, we need to recast the Vlasov equation in the polar coordinates previously defined in order to simplify its structure

$$\underbrace{\frac{\partial\psi}{\partial t} - \beta\eta c\delta\frac{\partial\psi}{\partial z} + \frac{\omega_s^2}{\eta\beta c}\frac{\partial\psi}{\partial\delta}}_{\omega_s\frac{\partial\psi}{\partial\Phi}} - F_w(z,t)\frac{\partial\psi}{\partial\delta} = 0$$

$$\text{from}\quad \frac{\partial\psi}{\partial\Phi} = \frac{\partial\psi}{\partial z}\frac{\partial z}{\partial\Phi} + \frac{\partial\psi}{\partial\delta}\frac{\partial\delta}{\partial\Phi} = -r\sin\phi\frac{\partial\psi}{\partial z} + r\cos\phi\frac{\partial\psi}{\partial\delta} =$$

$$= -\frac{\eta\beta c}{\omega_s}\delta\frac{\partial\psi}{\partial z} + \frac{\omega_s}{\eta\beta c}z\frac{\partial\psi}{\partial\delta}$$

$$\frac{\partial\psi}{\partial t} + \omega_s\frac{\partial\psi}{\partial\Phi} - \frac{F_w(z,t)}{p_0}\frac{\partial\psi}{\partial\delta} = 0$$

Linearization of Vlasov equation

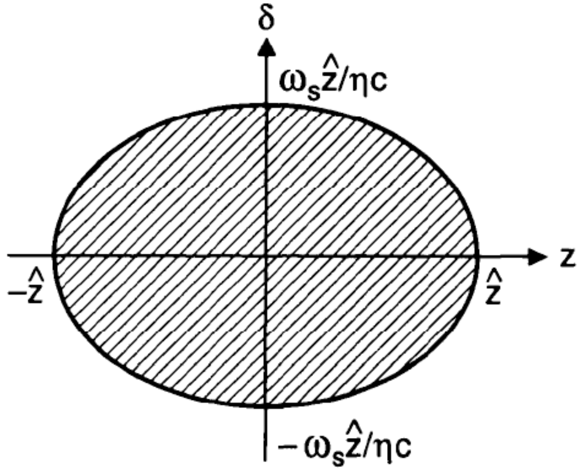
- We first substitute the ansatz on the previous slides for Ψ(r,Φ,t), and then we use the periodicity in the Φ coordinate to expand both sides of the equation in Fourier series
- Next, we specify to the case of $\Psi_0(r)$ being of water-bag type

$$-i\Omega\psi_1 + \omega_s \frac{\partial \psi_1}{\partial \Phi} - \frac{e^2 \eta c}{\omega_s E_0 T_0^2} \frac{1}{\sin\Phi} \frac{d\psi_0}{dr}$$

$$\times \sum_{p=-\infty}^{\infty} \exp\left[i(p\omega_0 + \Omega)\frac{r\cos\Phi}{c}\right] \tilde{\lambda}_1(p\omega_0+\Omega) Z_0^{||}(p\omega_0+\Omega)$$

$$\psi_1(r,\Phi) = \sum_{l=-\infty}^{\infty} \alpha_l R_l(r) \exp(il\Phi)$$

$$\psi_0(r) = \begin{cases} \dfrac{N_b \eta c}{\pi \hat{z}^2 \omega_s} & \text{if} \quad r < \hat{z} \\ 0 & \text{if} \quad r > \hat{z} \end{cases}$$



Linearization of Vlasov equation

- Now we are ready to equal the Fourier coefficients (index l) individually from both sides. This yields, in the limit of weak intensity beam, an equation for the (complex) frequency shifts associated to each mode.
- When an observer (impedance) sees the component $\lambda_1^{(l)}(z)$, the beam is executing the l-th mode of oscillation

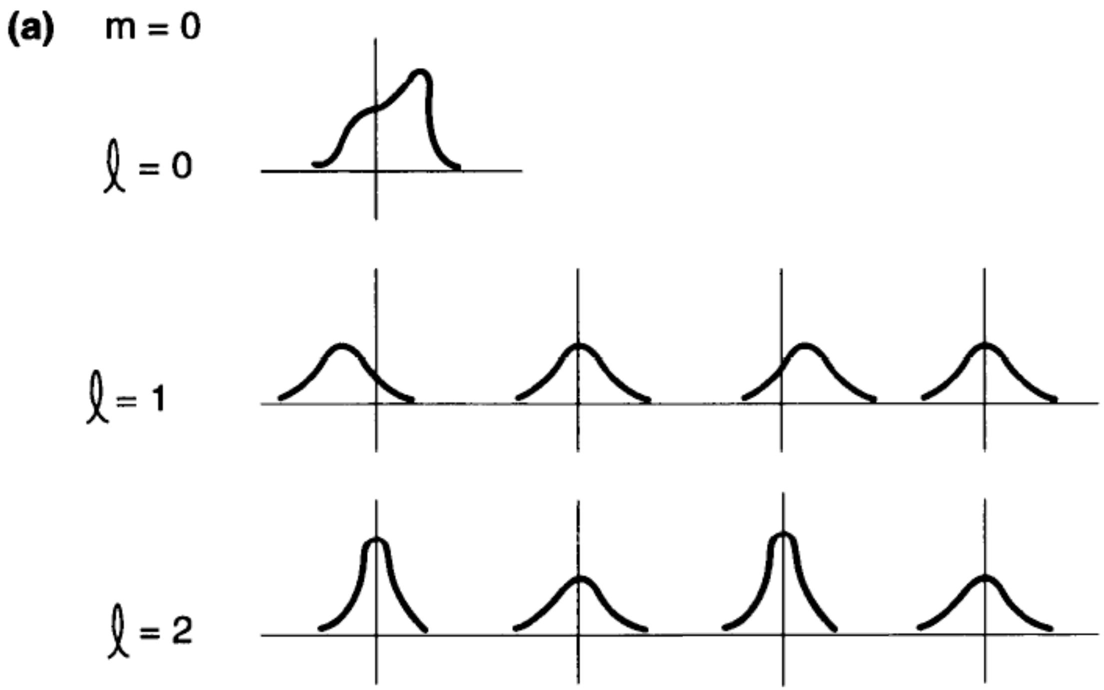
$$\Omega^{(l)} - l\omega_s = i\frac{lN_b e^2 \eta c^2}{2E_0 T_0^2 \omega_s \hat{z}^2} \sum_{p=-\infty}^{\infty} \frac{Z_0^{||}(p\omega_0 + l\omega_s)}{p\omega_0 + l\omega_s} \mathrm{J}_l^2 \left[\frac{(p\omega_0 + l\omega_s)\hat{z}}{c} \right]$$

- The imaginary part of the frequency shift gives the growth (or damping) rate of the mode. It only depends on the real part of the impedance
- For a broad band impedance the summation tends to an integral. Since the real part of the impedance is an even function of the frequency, as well as J_l^2, in this case the growth rate of all modes tends to zero because the function under summation (integral) is an odd function of frequency.

→ Short-lived wake fields (broad-band impedances) do not make beams longitudinally unstable

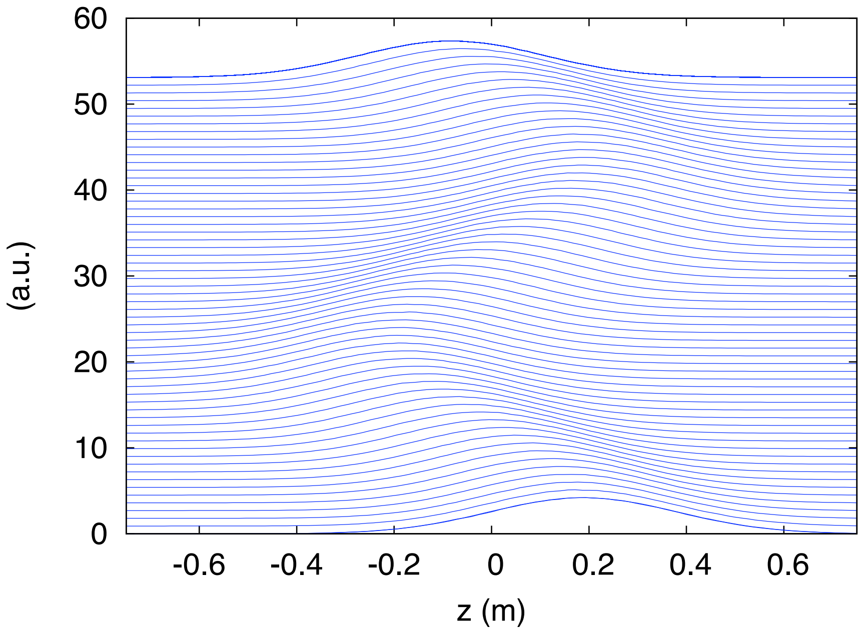
Linearization of Vlasov equation

- The modes given by the index l oscillate at frequencies close to $l\omega_s$.
- Since they have been introduced in relation with the azimuthal structure of the longitudinal phase space, they are referred to as longitudinal azimuthal modes
- From their definition and their oscillation frequencies, they can easily be associated to well distortion (l=0), dipole oscillation (l=1), quadrupole oscillation (l=2), etc.

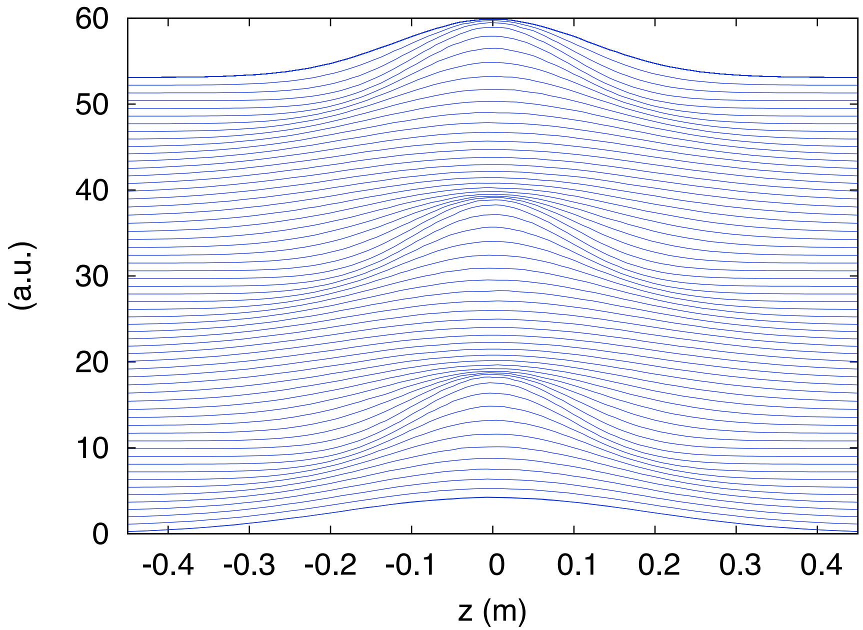


Linearization of Vlasov equation

1. (m=0 and l=1)



2. (m=0 and l=2)



Seen at a wide band pick up that can resolve the bunch (e.g. a wall current monitor)
These are mountain range plots showing the evolution in time of the bunch shape

1. The bunch is executing a rigid dipole oscillation in the longitudinal phase space at the synchrotron frequency
2. The bunch is mismatched in the bucket and it executes a quadrupole oscillation in the longitudinal phase space at twice the synchrotron frequency

The Robinson instability

- An interesting (and usual) case is when the impedance is peaked around a value $h\omega_0$ (impedance of the main rf cavity), because only the two terms $p=\pm h$ in the summation will contribute to the growth rate
- We also take the short bunch approximation (anyway the wake field lasts much longer than the bunch, being the impedance narrow-band)

$$\mathrm{J}_l\left(\frac{h\omega_0\hat{z}}{c}\right) \rightarrow \frac{1}{l!}\left(\frac{h\omega_0\hat{z}}{2c}\right)^l \quad \text{for} \quad \frac{h\omega_0\hat{z}}{c} \ll 1$$

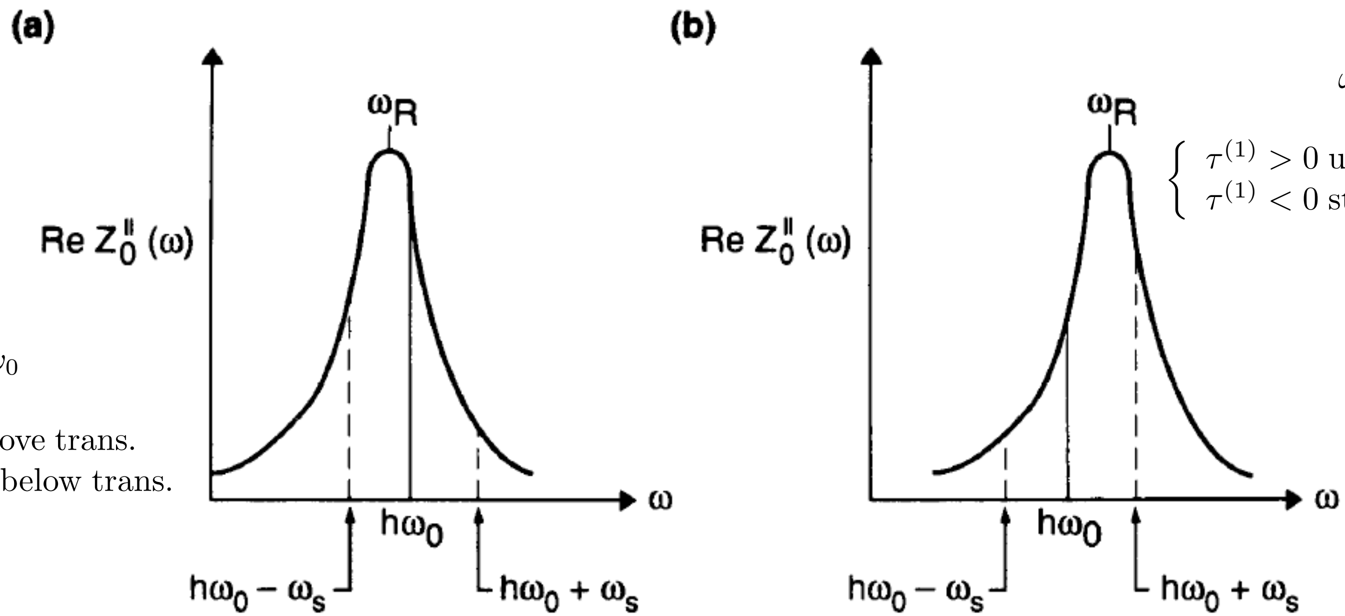
$$\frac{1}{\tau^{(l)}} = \frac{l}{(l!)^2}\frac{N_b e^2\eta c^2 h\omega_0}{2E_0T_0^2\omega_s}\left(\frac{h\omega_0\hat{z}}{c}\right)^{2l-2}\underbrace{\left[\mathrm{Re}\,Z_0^{||}(h\omega_0+l\omega_s)-\mathrm{Re}\,Z_0^{||}(h\omega_0-l\omega_s)\right]}$$

Instability or not, depends on the sign of this part…. Reminds: $d\mathrm{Re}(Z)/d\omega(h\omega_0)$

The Robinson instability

- The unstable modes that can be triggered by the rf cavities are called Robinson modes
- The most unstable one is the fundamental mode l=1, which is found also with a simple one particle approach.

$$\frac{1}{\tau^{(1)}} = \frac{N_b e^2 \eta c^2 h\omega_0}{2E_0 T_0^2 \omega_s} \left[Re\, Z_0^{||}(h\omega_0 + \omega_s) - Re\, Z_0^{||}(h\omega_0 - \omega_s) \right]$$



$$\omega_R < h\omega_0 \quad \Downarrow \quad \begin{cases} \tau^{(1)} < 0 \text{ stable above trans.} \\ \tau^{(1)} > 0 \text{ unstable below trans.} \end{cases}$$

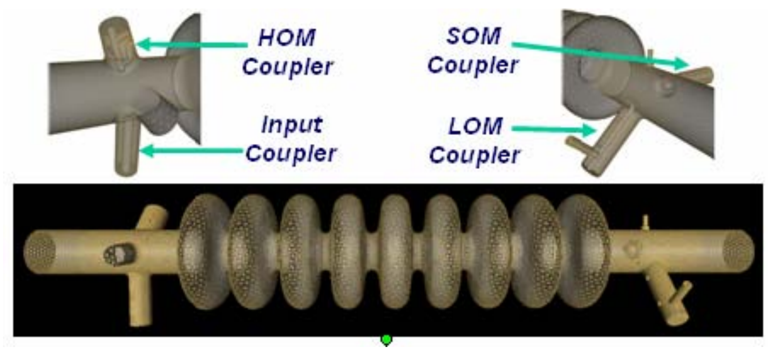
$$\omega_R > h\omega_0 \quad \Downarrow \quad \begin{cases} \tau^{(1)} > 0 \text{ unstable above trans.} \\ \tau^{(1)} < 0 \text{ stable below trans.} \end{cases}$$

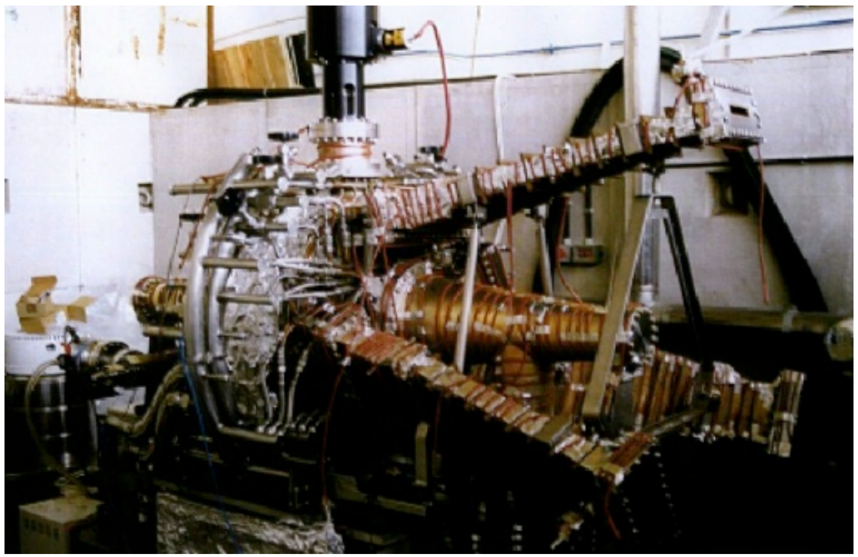
The Robinson instability

- Physically, the fundamental Robinson instability comes from the fact that the revolution frequency of an off-momentum beam is actually $\omega_0(1-\eta\delta)$. If the beam has an energy error, it will start a dipole oscillation and circulate at a frequency slightly higher than nominal when $\eta\delta<0$, or slightly lower when $\eta\delta>0$. Only if the beam samples the resistive part of the impedance such that it loses more energy when $\delta>0$ than when $\delta<0$, the motion will be damped.
- Although the Robinson instability was originally considered for the fundamental mode of the rf cavities, obviously the same analysis equally applies to all higher order rf modes
  - Attention must be paid that none of the $p\omega_0$ lands accidentally on the wrong side of some higher order impedance peak
  - Since there are typically several higher order modes in a cavity, and their tuning could be difficult to control, it is sometimes better to rely on a design of the cavity that damps these modes.
- The relative tuning of cavities with respect to the beam nominal revolution frequency is therefore yet one more thing that needs to be carefully controlled upon crossing transition in an accelerator…. (beside the rf phase and the sign of chromaticity)

The Robinson instability
HOMs in cavities

- To avoid getting into troubles with High Order Modes (HOMs) due to Robinson instabilities (or other types, too), usually cavities are equipped with HOM absorbers, purposely designed to absorb and suppress all the HOMs in the cavities
- Special purpose cavities, like crab cavities for example, which are not required to operate routinely on their fundamental mode, may need also LOM and SOM absorbers.
- Recently, the so called “photonic band” cavities are under investigation: through a special transverse arrangement (array-like structure), they have a very peaked resonance on the fundamental mode and naturally have all the HOMs flow away.





Radial modes

- Having assumed a water bag model for the unperturbed distribution, it turned out that one index alone would be sufficient to describe the collective motion, which we called l and was introduced in relation with the azimuthal structure in the longitudinal phase space
- This was an extremely simplified situation. For a general distribution, two longitudinal indices are needed. We call them l and n, where n is newly introduced to describe also the radial structure in the longitudinal phase space.
- Even with a distribution that highlights the radial structure of the longitudinal phase space, in the limit of zero beam intensity, all radial modes having the same azimuthal index l are degenerate and correspond to the same mode frequency $\Omega = l\omega_s$. As the beam intensity is increased slightly, their frequencies shift away from this unperturbed value and modes with different n's will move differently, breaking the previous degeneracy.

Radial modes

- Oscillation frequencies of the radial modes for different azimuthal families can be calculated for parabolic and Gaussian beams, for example
- Although not rigorous, the formulae here below are used in practice to establish synchrotron tune shifts and growth rates of the most prominent radial modes for each azimuthal mode

$$\Omega^{(l)} - l\omega_s = iF(l)\frac{\Gamma(l+1/2)}{(l-1)!}\frac{N_b e^2 \eta c^3}{E_0 T_0 \omega_s}\frac{\sum_{p=-\infty}^{\infty}\frac{Z_0^{||}(\omega')}{\omega'}h_l(\omega')}{\sum_{p=-\infty}^{\infty}h_l(\omega')}$$

Effective impedance

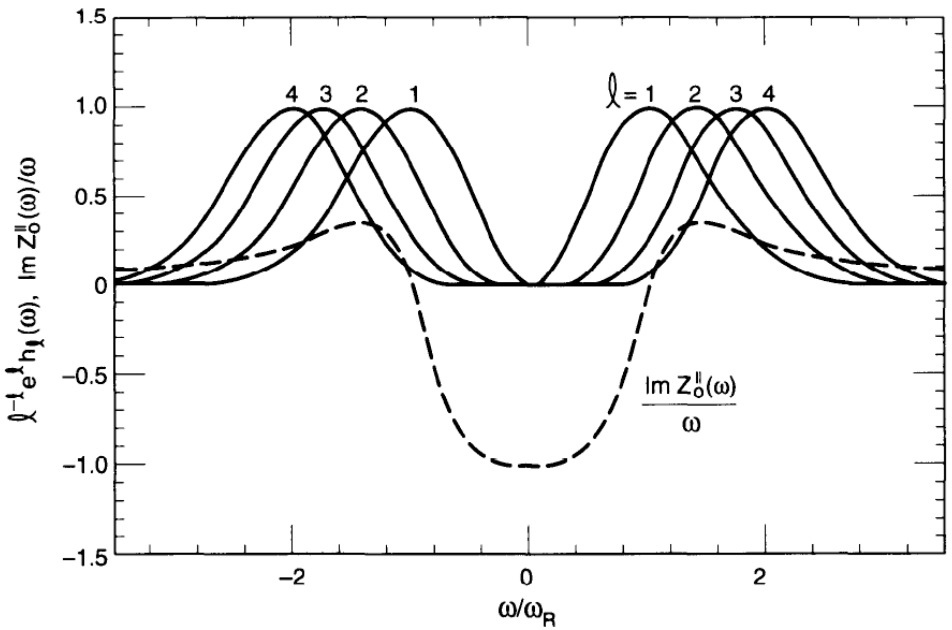
$$\omega' = p\omega_0 + l\omega_s$$

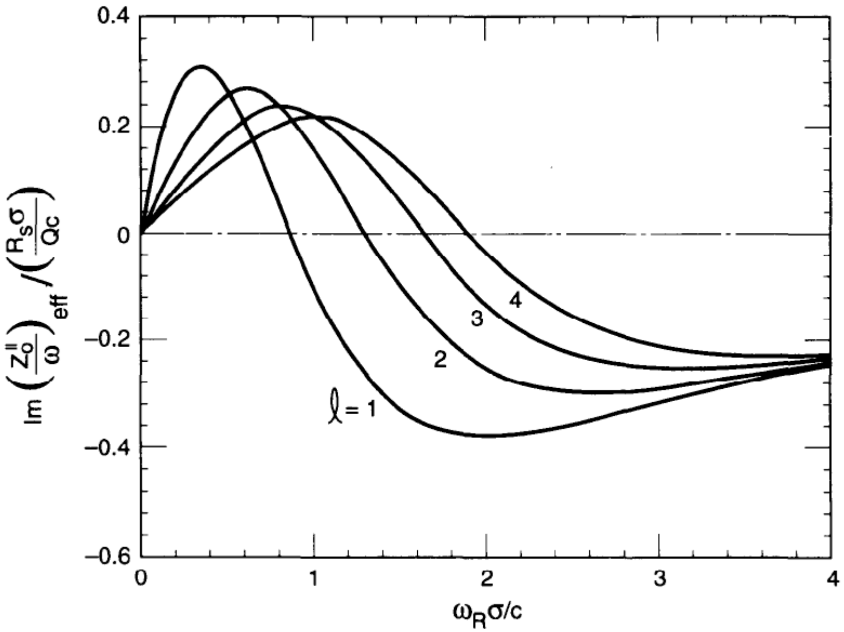
$$h_l(\omega) = \frac{\left[J_{l+1/2}(\omega\hat{z}/c)\right]^2}{|\omega\hat{z}/c|} \quad \text{parabolic} \qquad F(l) = \frac{3}{2\sqrt{\pi}\hat{z}^3} \quad \text{parabolic}$$

$$h_l(\omega) = \left(\frac{\omega\sigma_z}{c}\right)^{2l}\exp\left(-\frac{\omega^2\sigma_z^2}{c^2}\right) \quad \text{Gaussian} \qquad F(l) = \frac{1}{2^{l+1}\pi\sigma_z^3} \quad \text{Gaussian}$$

Radial modes

- For each mode, real and imaginary part of the coherent frequency shifts depend on the effective impedance associated to the mode
- For a broad-band resonator, the effective impedance is purely imaginary because the real part of the impedance is an even function. Again, we find the result that a broad-band impedance cannot make a weak beam unstable, not even when we consider radial modes in the analysis





Mode coupling

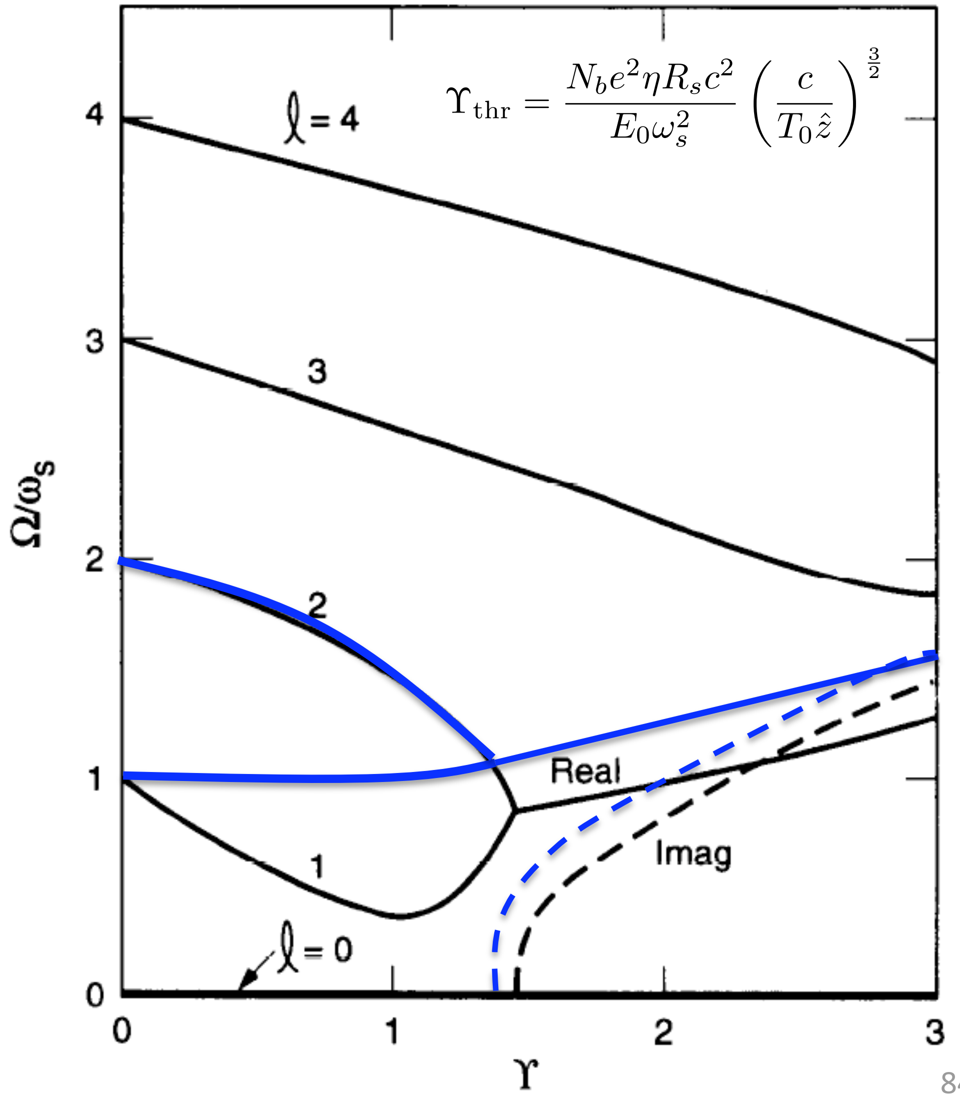
- All the previous analyses have been made for low intensity beams, and we always found that instability can only occur when the impedance consists of sharp peaks, or equivalently, when the wake fields last longer than the revolution period (Robinson type)
- The previous statement still holds even when radial modes are included in the analysis
- The reason is that, when the considered beam intensity is weak, the mode frequency shifts are small compared with the ω_s and hence, coupling between modes belonging to different azimuthal families could be excluded.
- When the beam intensity is increased and the frequency shifts become comparable to ω_s, possible coupling among modes must be taken into account, which can give rise to a different type of instability than the Robinson one.
- These instabilities are referred to as “microwave”, “mode coupling”, “mode mixing”, “turbulence” -type

Mode coupling

- Mode coupling is first investigated with an unperturbed water bag distribution and a broad-band impedance.
- As we know, for each azimuthal family, all radial modes converge together due to the radial degeneracy of the water bag distribution.
- For each azimuthal index l, the mode shift can be expressed as a function of a parameter Υ, proportional to the bunch intensity
- For a broad-band impedance model (shunt impedance R_s), it is found that there exists a threshold value Υ_{thr} above which two adjacent modes merge into one, meaning that two previously real solutions become a pair of complex conjugates, one damped and the other unstable
- Therefore, Υ_{thr} determines the bunch intensity above which an instability sets in. This instability is clearly a high intensity effect and is of a different nature than the one developed at lower beam intensities for sharply peaked impedances.

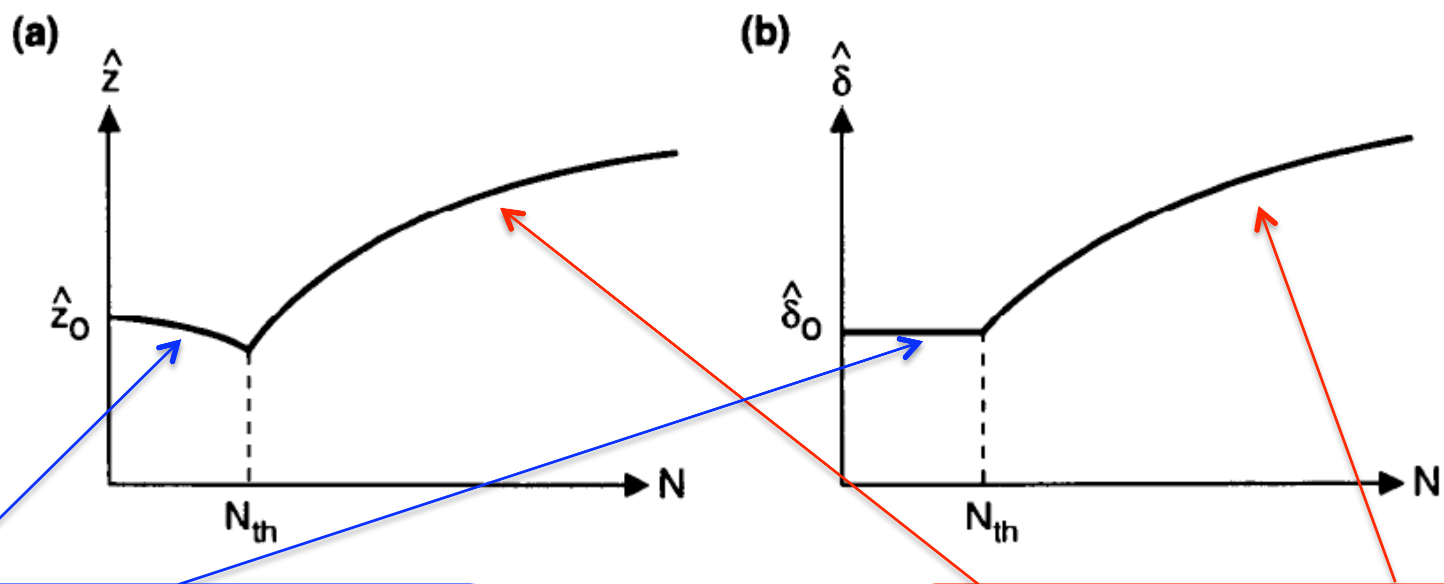
Mode coupling

- Mode 0 does not shift, because it corresponds to the potential well distortion. Mode 1 should not shift either, because it is related to rigid dipole motion and the wake field moves with the beam
- The reason why a shift appears in the plot on the right is that the term that would cancel it, i.e. a potential well distortion term, is neglected in the mathematical treatment. The **physical behavior of mode 1** is shown
- Unlike in the transverse plane, since modes 0 and 1 do not move, mode coupling instability in the longitudinal plane always involves modes with l≥2



Turbulent bunch lengthening

- The mode coupling instability is usually a non-destructive instability.
- In fact, as the intensity threshold for this instability is crossed, the beam becomes unstable and lengthens, causing the parameter Υ (inversely proportional to the 3/2 power of the bunch length) to fall back below the threshold value.

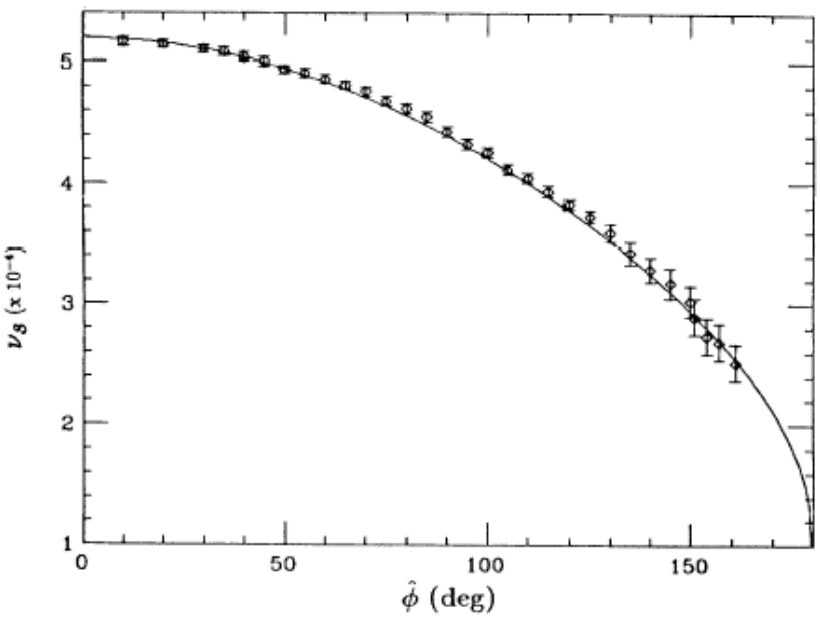


Potential well distortion below mode coupling threshold: example for an electron machine, with bunch shortening and constant momentum spread

$$\hat{z} = \frac{c}{T_0}\left(\frac{N_b e^2 \eta R_s c^2}{E_0 \omega_s^2 \Upsilon_{\rm thr}}\right)^{\frac{2}{3}}$$

Stability of bunches in realistic rf systems

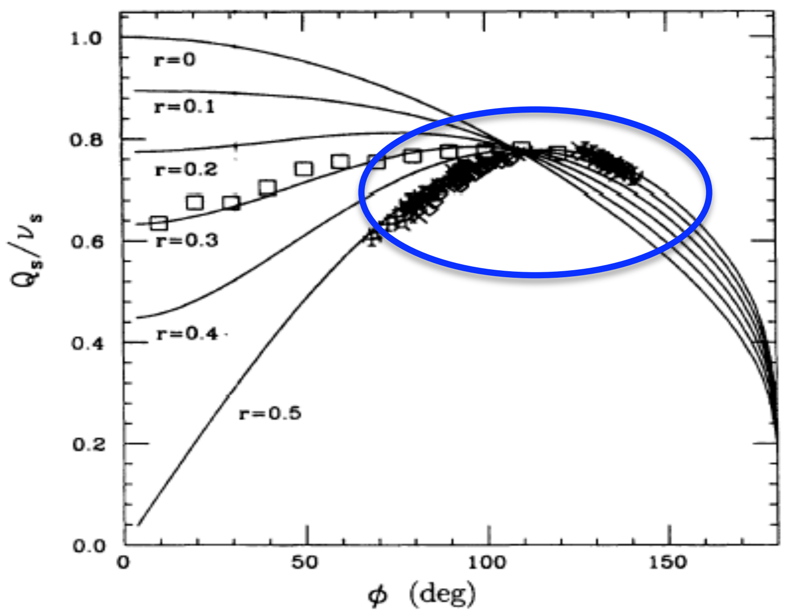
- Until now, we made an assumption on the Hamiltonian of the system, which limits the applicability of all results to the case of linear restoring force.
- The case of the stability of bunches in realistic rf systems (nonlinear) has been also studied in detail –at least for single and double sinusoidal voltage in the accelerating and stationary cases.
- The main general result of this study is that the nonlinearity of the force causes a spread in the synchrotron tune distribution, which helps stabilize the beam through Landau damping. The instability rates predicted by the case with linearized terms are therefore the "worst-case" ones.



Calculated and measured synchrotron tunes for particles having different synchrotron amplitudes (case of single harmonic cavity)

Stability of bunches in double harmonic rf systems

- The stability of bunches in a double harmonic rf system (accelerating and stationary bucket) has been studied.
- The main result is that the stability is found to depend on the derivative of the distribution of synchrotron tunes for particles at different amplitudes in the bucket:
  - Landau damping is lost in regions in which the derivative of the distribution of Q_s goes to zero. Beam current perturbations are enhanced and a microwave-type instability sets in.
  - Therefore, the single harmonic bucket is intrinsically stable, but when higher harmonic cavities are included, sensitive regions of the bunch are created (in fact we fall locally in the "worst case" mentioned on the previous slide).



Calculated and measured synchrotron tunes for particles having different synchrotron amplitudes (case of double harmonic cavity)

Coasting (unbunched) beams

- For an unbunched beam, the previously used equations of longitudinal motion still hold, assuming that $V_{rf}(z)=0$ and the only driving term in the momentum equation comes from space charge and wake fields

$$\begin{cases} \dot{z} = -\eta\beta c\delta \\ \dot{\delta} = \dfrac{1}{p_0}F_w(z,t) \end{cases}$$

$$\frac{\partial\psi}{\partial t} - \eta\beta c\frac{\partial\psi}{\partial z} + \frac{F_w(z,t)}{p_0}\frac{\partial\psi}{\partial\delta} = 0$$

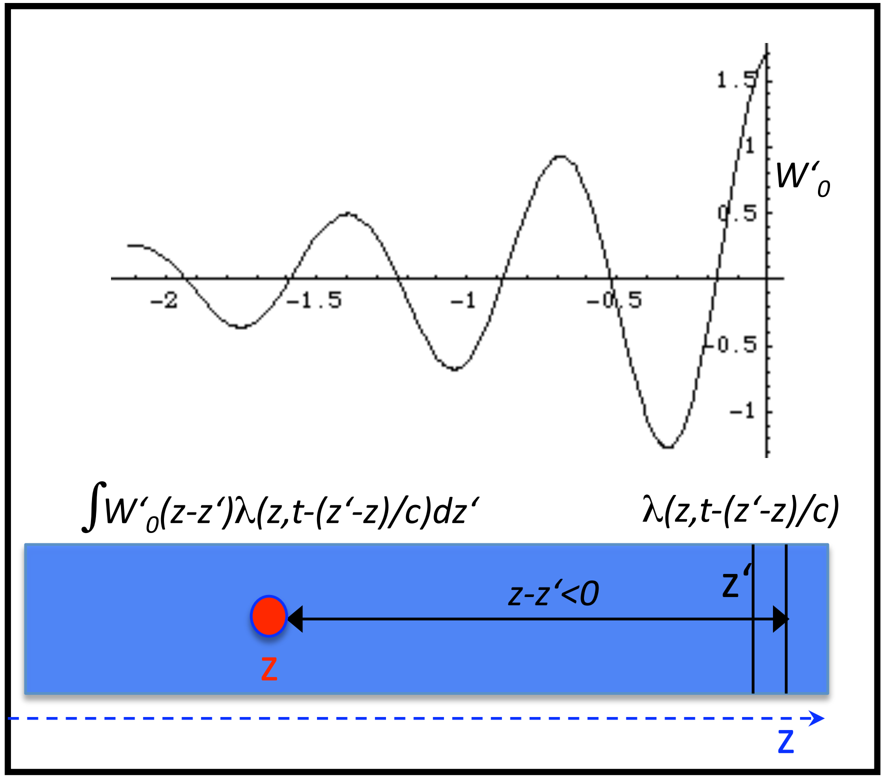
Given the unbunched and periodic nature of the solution, the unperturbed distribution will be only function of δ and we express the z-dependence of the perturbation through an angle variable

$$\psi(z,\delta,t) = \psi_0(\delta) + \psi_1(\delta)\exp[i(n\vartheta - \Omega t)]$$

$$\begin{cases} \displaystyle\int_{\Re}\psi_0(\delta)d\delta = \frac{N_b}{2\pi R} \\ \vartheta = \bar{\omega}_0 t + \dfrac{z}{R} \end{cases}$$

Coasting (unbunched) beams

$$F_w(z,t) = -\frac{e^2}{C}\int_{-\infty}^{\infty} \lambda\left(z\ ,t - \frac{z'-z}{c}\right) W_0'(z-z')dz'$$



- We change perspective: a charge located in z at time t feels the effect of the z' slice through the wake generated in z by the line density at a time $\Delta t=(z'-z)/c$ earlier
- The constant part λ_0 (not a function of z) only contributes with a constant term, which is a net energy loss if $\int W'_0(z)dz>0$, or equivalently $\mathrm{Re}[Z_0^{||}(\omega=0)]>0$, (i.e. the beam produces dc losses)
- The time varying part of $\lambda(z,t)$ generates a retarding field potentially responsible for instabilities

Coasting (unbunched) beams

$$F_w(\vartheta,t) = -\frac{e^2}{C}\int_{\Re}\psi_1(\delta)d\delta \times \int_{\Re}\exp(in\vartheta)\exp\left[-i\Omega\left(t-\frac{R\vartheta'-R\vartheta}{c}\right)\right]W_0'(R\vartheta-R\vartheta')Rd\vartheta'$$

$$F_w(\vartheta,t) = -\frac{ce^2}{C}\exp(in\vartheta-\Omega t)Z_0^{||}(\Omega)\int_{\Re}\psi_1(\delta)d\delta$$

- We easily recast the wake field driving term in terms of impedance (by simply using the definition of impedance), and then substitute it in the Vlasov equation

$$(-i\Omega+in\bar{\omega}_0)\psi_1 - in\eta\beta c\delta\frac{1}{R}\psi_1 - \frac{ce^2Z_0^{||}(\Omega)}{Cp_0}\frac{d\psi_0}{d\delta}\int_{\Re}\psi_1(\delta)d\delta = 0$$

Coasting (unbunched) beams

$$\psi_1(\delta) = i\frac{e^2 c Z_0^{||}(\Omega)\psi_0'(\delta)}{\beta^2 E_0 T_0[\Omega - \bar{\omega}_0 n(1-\eta\delta)]}\int_{\Re}\psi_1(\delta')d\delta'$$

$$1 = i\frac{e^2 c Z_0^{||}(\Omega)}{\beta^2 E_0 T_0}\int_{\Re}\frac{\psi_0'(\delta)}{\Omega - \bar{\omega}_0 n(1-\eta\delta)}d\delta$$

We now change from the variable δ to the variable ω_0, and we prefer to have a distribution function normalized to unity

$$\omega_0 = \bar{\omega}_0(1-\eta\delta)$$

$$\rho(\omega_0) = \frac{C}{N|\eta|\bar{\omega}_0}\psi_0(\delta) \;\Rightarrow\; \rho'(\omega_0) = -\frac{C}{N_b\eta|\eta|\bar{\omega}_0^2}\psi_0'(\delta)$$

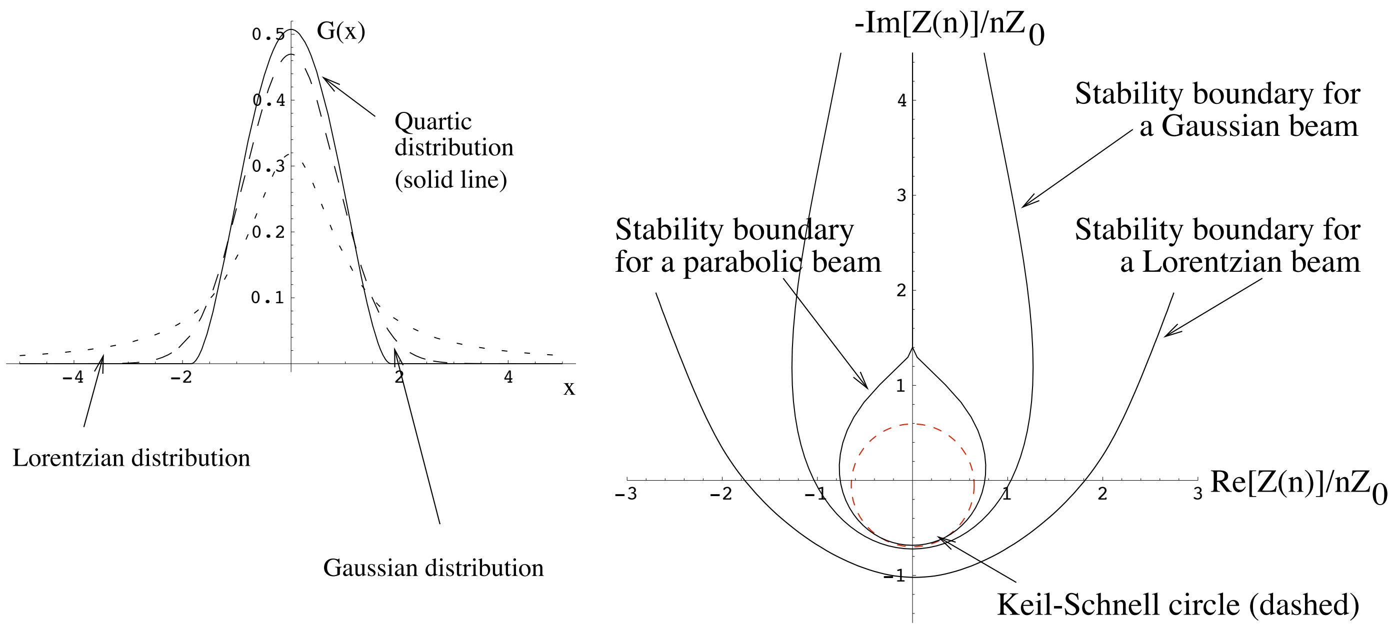
Coasting (unbunched) beams

Dispersion relation

$$1 = -i\frac{2\pi N\eta e^2 Z_0^{||}(\Omega)}{\beta^2 E_0 T_0^3}\int_{\Re}\frac{\rho_0'(\omega_0)}{\Omega - n\omega_0}d\omega_0$$

- If we specify the momentum (or equivalently, revolution frequency) distribution, the previous dispersion relation can be used to find the solutions Ω, expressing the frequency shift
- Complex solutions with positive imaginary part indicate instability
- The use of the dispersion relation is often to consider $Z_0^{||}(\Omega)$ as a free complex number and then draw in the complex (impedance) plane the curve Im[Ω]=0, which is the stability boundary separating the stable region (Im[Ω]<0) from the unstable region (Im[Ω]>0)
- Then care must be taken that the impedance of the machine falls within the stability region
- This approach is usually considered valid both for coasting beams and for long bunched beams with slow synchrotron motion (Boussard criterion)

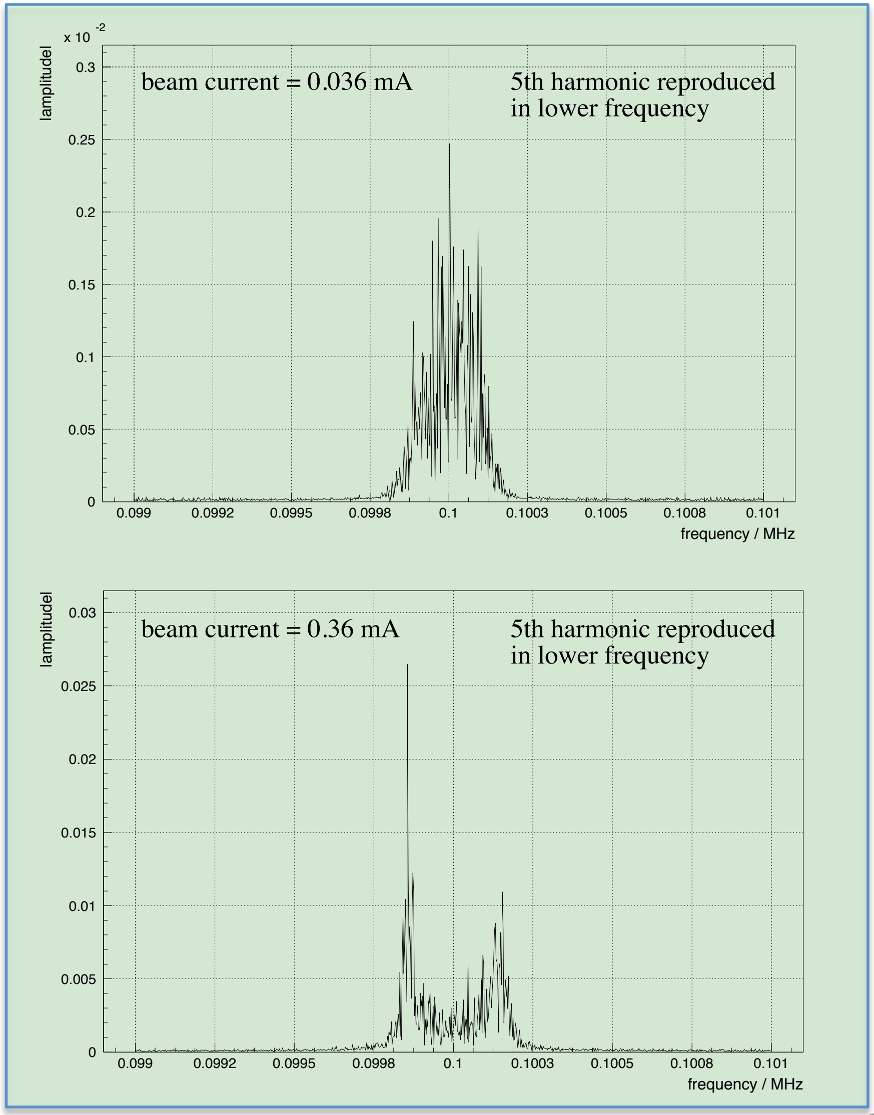
Coasting (unbunched) beams



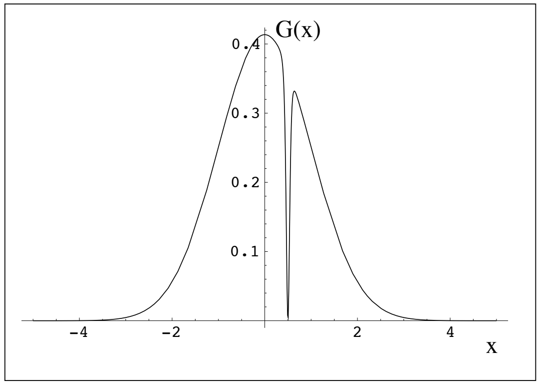
- **Exercise:** Show that, if the beam is monochromatic (i.e., all particles circulate with the same revolution frequency ω_0), it is always unstable above transition with a capacitive impedance (e.g. space charge) and below transition with an inductive impedance. The instability with space charge above transition is simply a negative mass instability
- Different distributions have different stability regions, whose extension depends on the thickness of the tails of the distribution (see above). The mechanism behind the existence of a stability region is Landau damping (stabilization through spread of the frequencies), hence it is wider for larger spreads.

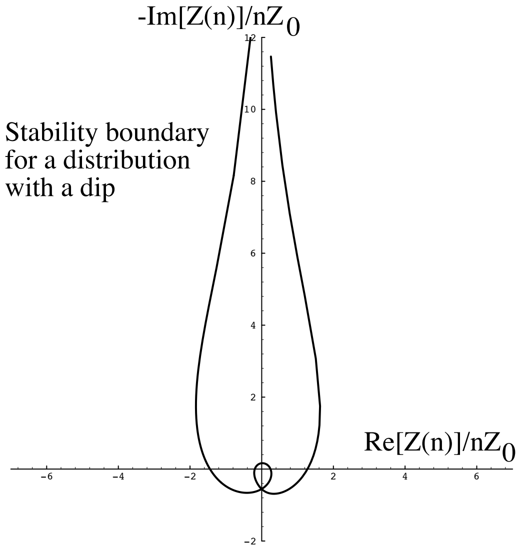
Coasting (unbunched) beams

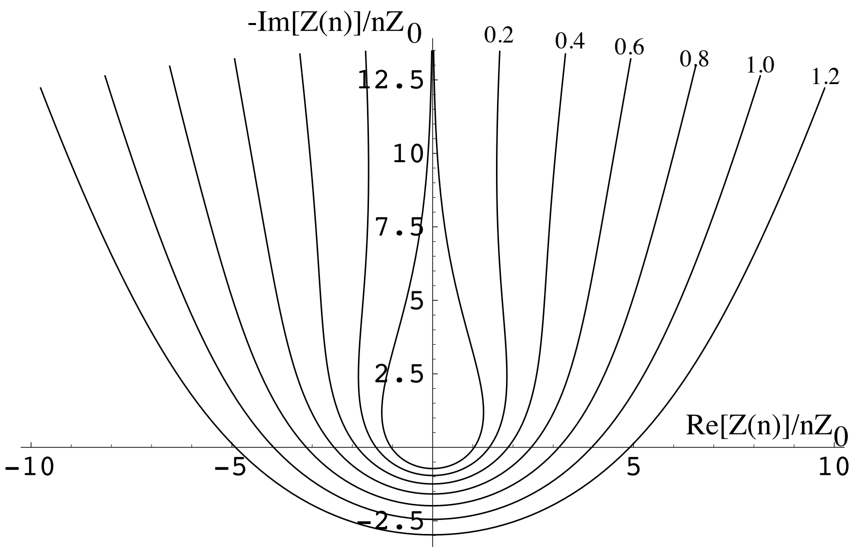
- It can be shown that, for a given impedance, the dispersion relation always has two solutions, with opposite signs of Re[Ω-nω_0], corresponding to a fast wave and a slow wave. Both are stable inside the stability region, whereas the slow wave is unstable outside of it (the fast wave is always stable)
- Fast and slow wave can be clearly seen in the Schottky spectrum of the beam, i.e. the Fourier transform of a wide-band pick up installed in a ring. When no significant coherent motion is present (i.e. for weak space charge and low impedance), the Schottky spectrum simply shows lines centered at multiples of ω_0 whose width can be related to the momentum spread of the beam.



Coasting (unbunched) beams







- A dip in the momentum distribution function strongly reduces the stability region. However, the instability only serves the purpose to fill the dip and create a stable distribution
- Beside the stability boundary, it is also useful to draw all curves Im[Ω]=k in the impedance plane, so that the growth time of an instability for a given impedance working point can be deduced from the instability chart.

Coasting (unbunched) beams

- If the detailed spectral information is not available and the dispersion relation cannot be solved in detail to determine a stability region, the simplified so called ‘Keil-Schnell’ criterion for stability can be used
- It is a handy formula and gives a rough estimate whether the beam is stable or not against microwave instability. F is a form factor always close to 1.
- It can be applied to bunched beams, as well, provided that the growth rate of the instability is much larger than the synchrotron frequency and the wavelength of the perturbation (~R/n) is much smaller than bunch length (~4σ_z)

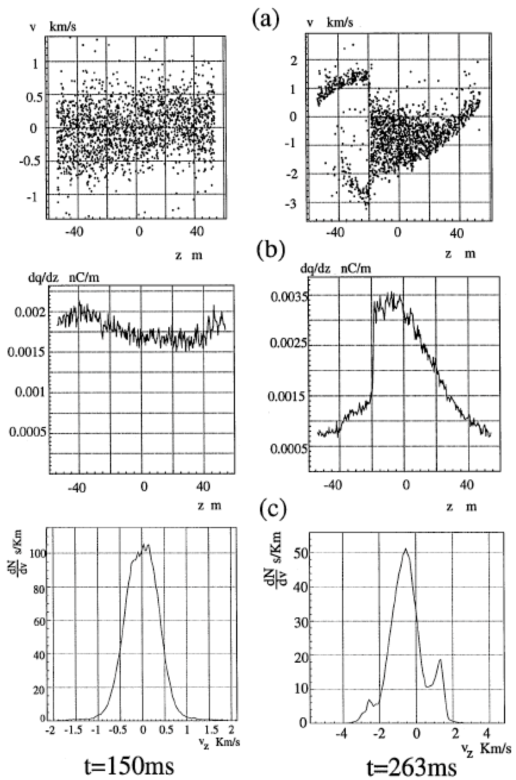
$$\frac{2\pi N e^2}{E_0 T_0^3} |n\eta Z_0^{||}(n\omega_0)| < F n^2 \Delta\omega^2$$

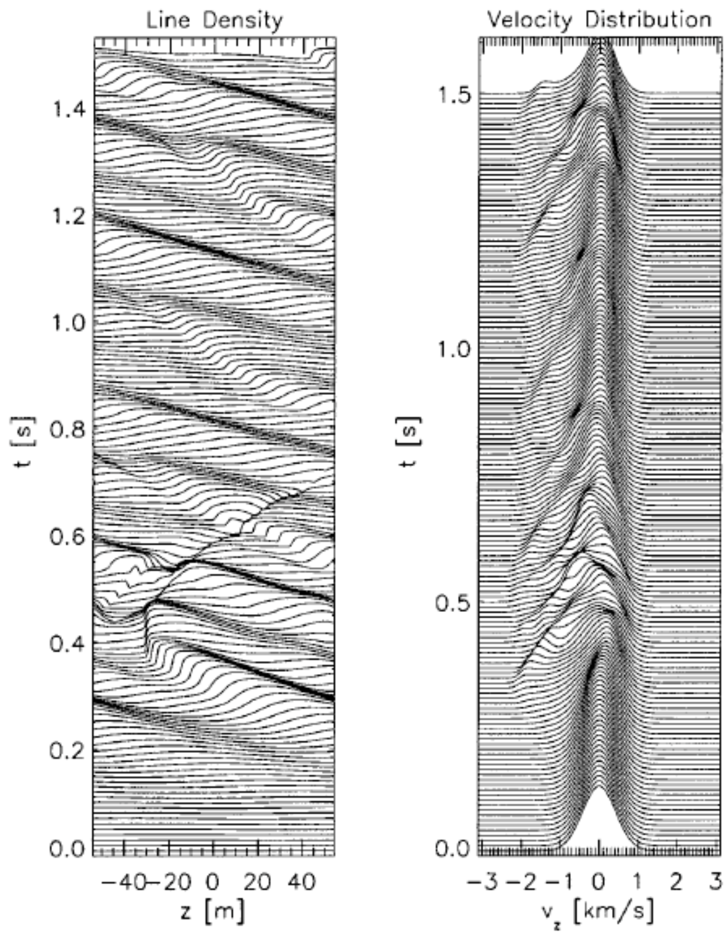
$$\left| \frac{Z_0^{||}(n\omega_0)}{n} \right| < F \cdot \frac{E_0|\eta|}{\bar{\omega}_0 N e^2} \left(\frac{\delta p}{p_0} \right)^2_{\mathrm{FWHM}}$$

Keil-Schnell criterion

Coasting (unbunched) beams

- The perturbative Vlasov approach is good to estimate stability regions and instability rise time, when the beam is expected to become unstable
- However, the longitudinal instability of coasting beams can be also solved numerically by using
  - → Macroparticle simulations
  - → Poisson-Vlasov solvers





Coasting (unbunched) beams

- The two main features that emerge from the nonlinear evolution are:
  - Nonlinear wave steepening leading to saturation and decay (formation of holes in phase space)
  - Momentum overshoot: when the beam reaches another steady state, the momentum spread is typically higher than the one required to stabilize the initial beam
- The longitudinal instability measured at the GSI-ESR shows an impressive resemblance to the evolution predicted with a Vlasov model