

# BUNCHED-BEAM TRANSVERSE COHERENT INSTABILITIES

- ◆ **(Single-bunch linear) Head-Tail phase shift (10 Slides)**
- ◆ **Vlasov formalism (7)**
- ◆ **Low intensity => Head-tail modes: (Slow) head-tail instability (43)**
- ◆ **High intensity => Coupling of the head-tail modes: Transverse Mode Coupling Instability (TMCI) or (Fast) head-tail instability (40)**
- ◆ **Transverse coupled-bunch instability in time domain (11)**

## HEAD-TAIL PHASE SHIFT (1/10)

- ◆ **Let's have a look first to the effect of chromaticity on the transverse bunch dynamics, as it is the key ingredient for instabilities**
- ◆ **Equation of motion for a single particle in longitudinal phase space (using polar coordinates) considering only the linear force and neglecting collective effects (see previous courses)**

$$z(s; r, \phi_s) = r \cos \phi_z$$

$$\frac{\eta C}{2\pi Q_s} \delta(s; r, \phi_s) = r \sin \phi_z$$

$$\phi_z = \frac{2\pi Q_s}{C} s + \phi_s$$

- ◆ **After a transverse kick the particle also undergoes transverse motion which, turn after turn ( $n = s / C$ ), can be described by**

$$y(n; r, \phi_s) = A \sin \left[ 2\pi n Q_y + \vartheta(n; r, \phi_s) \right]$$

## HEAD-TAIL PHASE SHIFT (2/10)

with, assuming a purely linear chromaticity,

$$\vartheta(n; r, \phi_s) = 2\pi \int_0^{nC} \Delta Q_y dk$$

$$k = \frac{s}{C}$$

$$\Delta Q_y = Q'_y \delta(s; r, \phi_s)$$

$$\Rightarrow \vartheta(n; r, \phi_s) = \frac{2\pi}{C} \int_0^{nC} ds Q'_y \delta(s; r, \phi_s)$$

$$\delta = -\frac{z'}{\eta}$$

$$\Rightarrow \vartheta(n; r, \phi_s) = -\frac{2\pi Q'_y}{\eta C} \left[ z(nC; r, \phi_s) - z(0; r, \phi_s) \right]$$

- ◆ This phase shift can then be expressed as a function of the actual position of the particle in the longitudinal phase space

$$\left( \tau \equiv \frac{z}{\beta c}, \delta \right)$$

## HEAD-TAIL PHASE SHIFT (3/10)

$$\begin{aligned}
 r \cos(\phi_s) &= r \cos(2\pi n Q_s + \phi_s - 2\pi n Q_s) \\
 &= z \cos(2\pi n Q_s) + \frac{\eta C}{2\pi Q_s} \delta \sin(2\pi n Q_s)
 \end{aligned}$$

$$\Rightarrow \vartheta(n; r, \phi_s) \equiv \vartheta(n; \tau, \delta) = -\frac{\Omega_0 Q'_y}{\eta} [1 - \cos(2\pi n Q_s)] \tau + \frac{Q'_y}{Q_s} \sin(2\pi n Q_s) \delta$$

$$\begin{aligned}
 \Rightarrow y(n; \tau, \delta) &= A \cos\left[\frac{Q'_y}{Q_s} \delta \sin(2\pi n Q_s)\right] \sin\left[2\pi n Q_y - \frac{\Omega_0 Q'_y}{\eta} \tau [1 - \cos(2\pi n Q_s)]\right] \\
 &+ A \sin\left[\frac{Q'_y}{Q_s} \delta \sin(2\pi n Q_s)\right] \cos\left[2\pi n Q_y - \frac{\Omega_0 Q'_y}{\eta} \tau [1 - \cos(2\pi n Q_s)]\right]
 \end{aligned}$$

## HEAD-TAIL PHASE SHIFT (4/10)

- ◆ At turn  $n$ , the transverse excursion  $\langle y \rangle(\hat{\tau}; n)$  of the slice  $\hat{z} \equiv \beta c \hat{\tau}$  can then be obtained by multiplying the previous relation by the actual longitudinal distribution  $\rho(\hat{\tau}, \delta; n)$  of the bunch, integrating over  $\delta$  and normalizing the result

$$\langle y \rangle(\hat{\tau}; n) = A(\hat{\tau}; n) \sin[2\pi n Q_y + \phi_y(\hat{\tau}; n)] + B(\hat{\tau}; n) \cos[2\pi n Q_y + \phi_y(\hat{\tau}; n)]$$

with

$$A(\hat{\tau}; n) = A \frac{\int d\delta \rho(\hat{\tau}, \delta; n) \cos\left[\frac{Q'_y}{Q_s} \delta \sin(2\pi n Q_s)\right]}{\int d\delta \rho(\hat{\tau}, \delta; n)}$$

$$B(\hat{\tau}; n) = A \frac{\int d\delta \rho(\hat{\tau}, \delta; n) \sin\left[\frac{Q'_y}{Q_s} \delta \sin(2\pi n Q_s)\right]}{\int d\delta \rho(\hat{\tau}, \delta; n)}$$

$$\phi_y(\hat{\tau}; n) = -\frac{\Omega_0 Q'_y}{\eta} \hat{\tau} [1 - \cos(2\pi n Q_s)]$$

## HEAD-TAIL PHASE SHIFT (5/10)

- ◆ After the RF capture, the distribution  $\rho$  becomes independent of  $n$  (assuming no coherent longitudinal oscillations) and is an even function of  $\delta$

$$\Rightarrow \langle y \rangle (\hat{\tau}; n) = A(\hat{\tau}; n) \sin \left[ 2\pi n Q_y + \phi_y(\hat{\tau}; n) \right]$$

- ◆ If we consider the evolution of 2 longitudinal positions within a single bunch separated in time by  $\Delta\tau$ , then the phase difference in the transverse oscillation of these 2 slices is given by

$$\Delta\phi_y(\Delta\tau; n) = -\frac{\Omega_0 Q'_y}{\eta} \Delta\tau \left[ 1 - \cos(2\pi n Q_s) \right]$$

- ◆ This phase difference is a maximum when  $n Q_s = \frac{1}{2}$ , i.e. after  $\frac{1}{2}$  a synchrotron period

## HEAD-TAIL PHASE SHIFT (6/10)

- ◆ It is directly related to the chromaticity by

$$\Delta\phi_y^{\max}(\Delta\tau) = -\frac{2\Delta\tau\Omega_0 Q'_y}{\eta}$$

or

$$Q'_y = -\frac{\eta\Delta\phi_y^{\max}(\Delta\tau)}{2\Delta\tau\Omega_0}$$

- ◆ Furthermore, there is also information related to the decoherence of the signal observed. Considering a Gaussian distribution, one has

$$\rho(\tau, \delta) = \frac{1}{2\pi\sigma_\tau\sigma_\delta} e^{-\frac{\tau^2}{2\sigma_\tau^2} - \frac{\delta^2}{2\sigma_\delta^2}}$$

with

$$\sigma_\delta = \frac{\Omega_0 Q_s}{|\eta|} \sigma_\tau$$

## HEAD-TAIL PHASE SHIFT (7/10)

⇒

$$A(\hat{\tau}; n) = A e^{-\left\{ \frac{\left[ \frac{\Omega_0 Q'_y}{\eta} \sigma_\delta \sin(2\pi n Q_s) \right]^2}{2} \right\}}$$

⇒ Therefore, in the presence of non-zero chromaticity, the signal envelope decoheres and recoheres every  $\frac{1}{2}$  synchrotron period

- ◆ Finally, the signal revealed by a transverse Beam Position Monitor in the control room of an accelerator is given by

$$\langle y \rangle (\hat{\tau}; n) \times \frac{1}{\sqrt{2\pi} \sigma_\tau} e^{-\frac{\hat{\tau}^2}{2\sigma_\tau^2}}$$

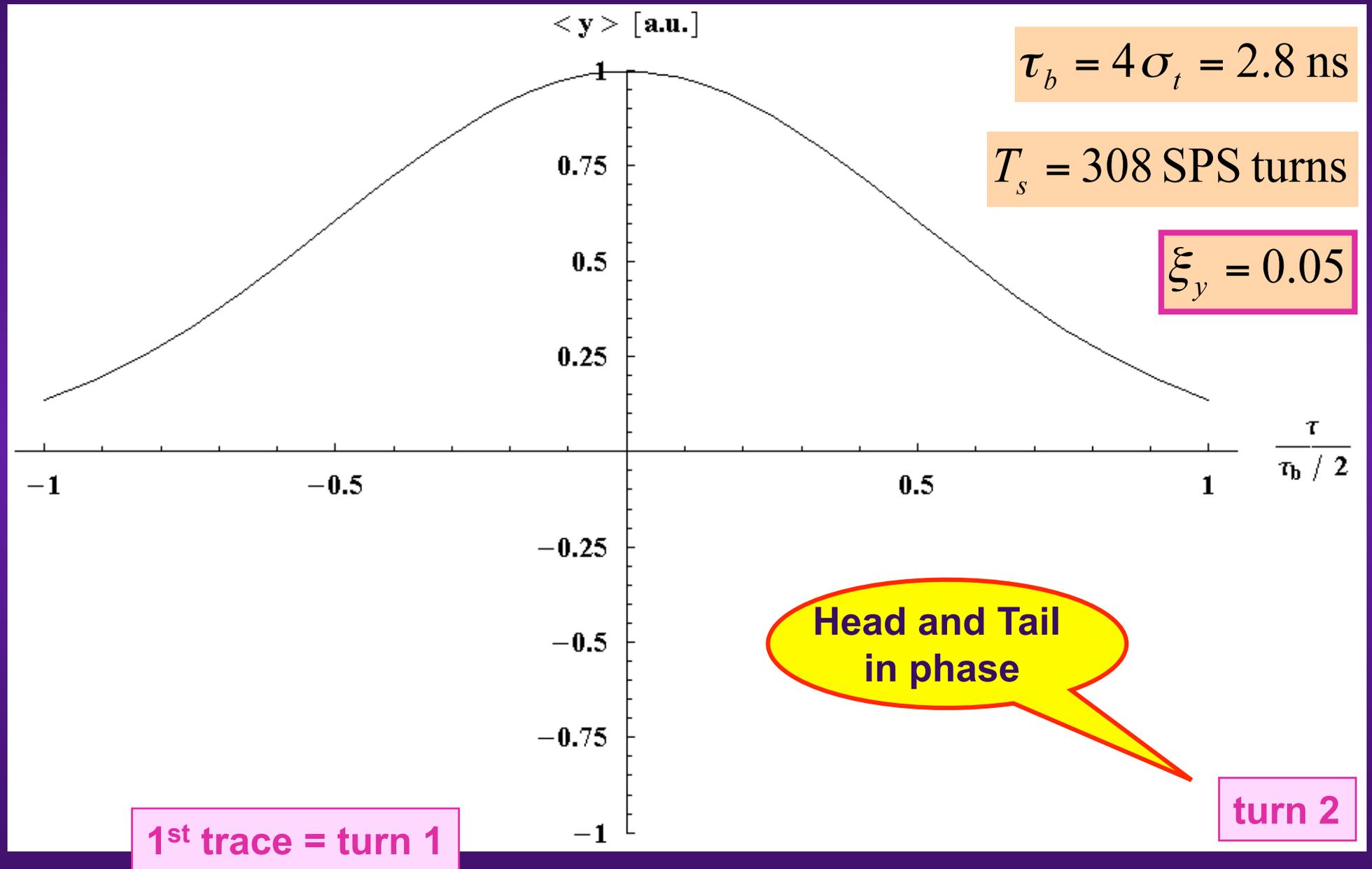
Transverse excursion of a slice

Longitudinal distribution

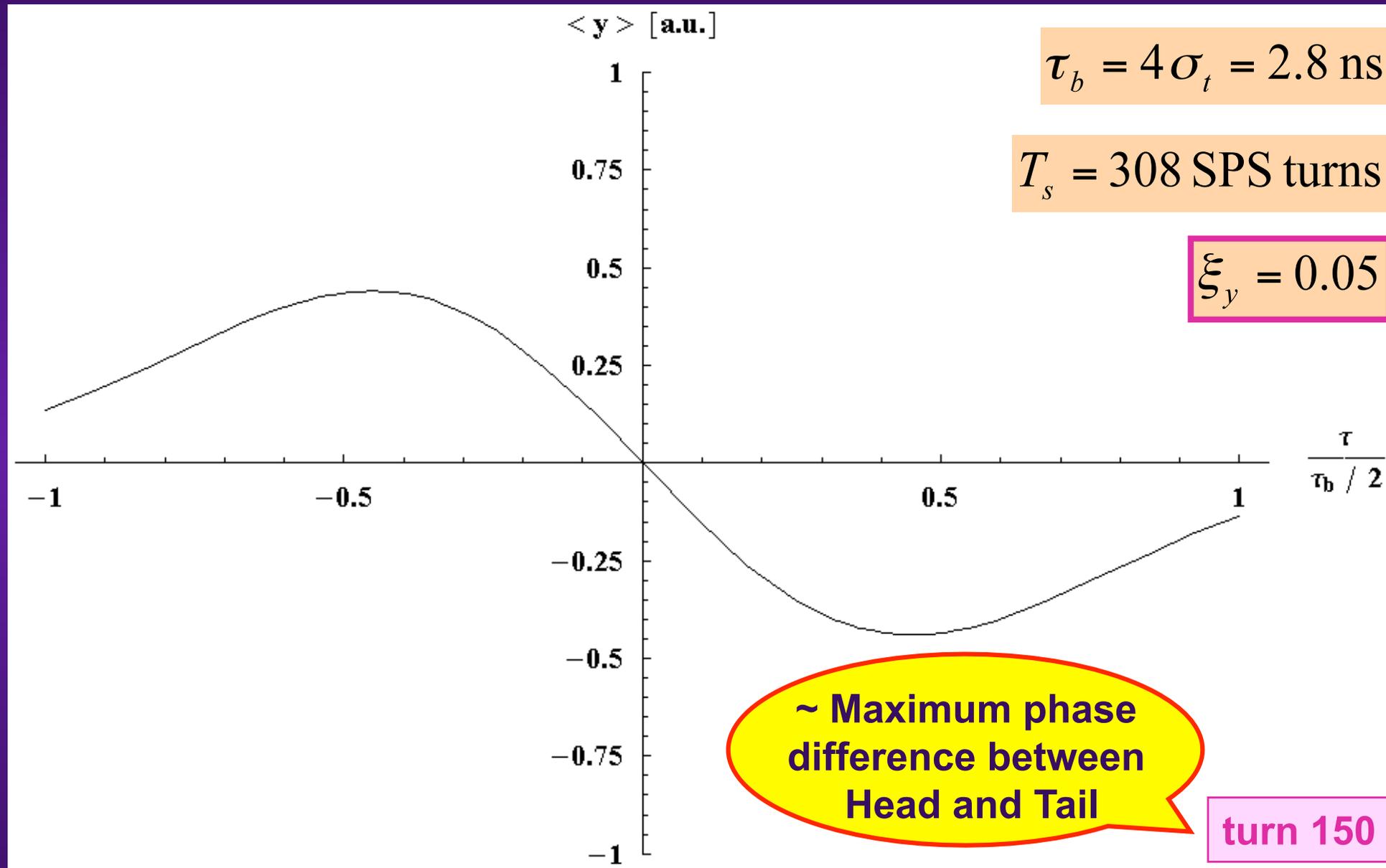
## HEAD-TAIL PHASE SHIFT (8/10)

**=> See the Movie for the case of a CERN SPS bunch for the LHC  
(under Windows!)**

# HEAD-TAIL PHASE SHIFT (9/10)



# HEAD-TAIL PHASE SHIFT (10/10)



# VLASOV FORMALISM (1/7)

## SINGLE-PARTICLE EQUATION FORMALISM

**Coupled-bunch modes**  
**Courant and Sessler**

$$n = 0, 1, \dots, M - 1$$

**Particular impedances and oscillation modes**

**Head-tail modes**  
**Pellegrini and Sands**

$$m = \dots, -1, 0, 1, \dots$$

**Generic impedances and high order head-tail modes**

**VLASOV FORMALISM**  
 $\Rightarrow$  **Distribution of particles**  
 $\Rightarrow$  **Liouville's theorem**

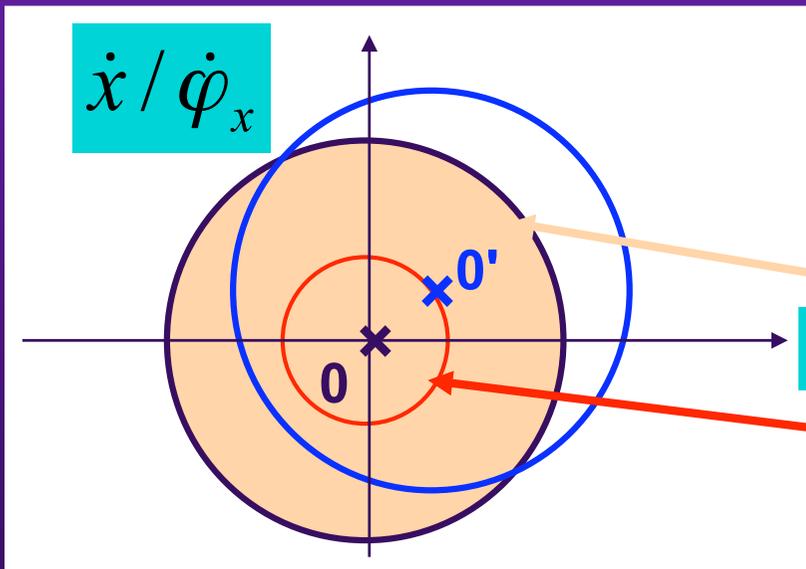
$\Rightarrow$  **Radial modes**

$$q = \dots, -1, 0, 1, \dots$$

- **Sacherer's** integral equation
- **Laclare's** eigenvalue problem

Horizontal stationary distribution

Coherent motion at  $\omega_c$



## VLASOV FORMALISM (2/7)

- ◆ The results discussed before for coasting beams can also be re-derived using the Vlasov formalism
- ◆ The basic mathematical tool used for the mode representation of the beam motion is the Vlasov equation, which describes the collective behaviour of a multiparticle system under the influence of electromagnetic forces
- ◆ It can be derived from the conservation of the phase-space area (as stated by the Liouville's theorem)
- ◆ To construct the Vlasov equation, one starts with the single-particle equations of motion
- ◆ The coordinates  $q_\rho$  and  $p_\rho$  (with  $\rho = x, y, z$ ) should be canonically conjugated, which means that they should be derived from a Hamiltonian  $H(q_\rho, p_\rho, t)$  by the canonical equations

$$\dot{q}_\rho = \frac{\partial H(q_\rho, p_\rho, t)}{\partial p_\rho}$$

$$\dot{p}_\rho = - \frac{\partial H(q_\rho, p_\rho, t)}{\partial q_\rho}$$

## VLASOV FORMALISM (3/7)

- ◆ According to the Liouville's theorem, the particles, in a non-dissipative system of forces, move like an incompressible fluid in phase space. The constancy of the phase space density  $\Psi(q_\rho, p_\rho, t)$  is expressed by the equation

$$\frac{d\Psi(q_\rho, p_\rho, t)}{dt} = 0$$

where the total differentiation indicates that one follows the particle while measuring the density of its immediate neighborhood. This equation, sometimes referred to as the Liouville's theorem, states that the local particle density does not vary with time when following the motion in canonical variables

## VLASOV FORMALISM (4/7)

- ◆ **Practically, one would like to know the development of this density as seen by a stationary observer (like a beam monitor) which does not follow the particle**
- ◆ **It depends now not only directly on the time but also indirectly through the coordinates of the moving particles, which change with time**

$$\Rightarrow \frac{\partial \Psi(q_\rho, p_\rho, t)}{\partial t} + \dot{q}_\rho \frac{\partial \Psi(q_\rho, p_\rho, t)}{\partial q_\rho} + \dot{p}_\rho \frac{\partial \Psi(q_\rho, p_\rho, t)}{\partial p_\rho} = 0$$

- ◆ **This expression is the Vlasov equation in its most simple form and is nothing else but an expression for the Liouville's conservation of phase-space density seen by a stationary observer**

## VLASOV FORMALISM (5/7)

- ◆ In Liouville's theorem the phase-space area is only conserved if expressed in canonically conjugated variables
- ◆ The same criterion applies to the validity of the Vlasov equation
- ◆ However, these variables are often not very practical for accelerator applications, and other coordinates are sometimes used in an approximate manner
- ◆ Strictly speaking,  $\dot{q}_\rho$  and  $\dot{p}_\rho$  are given by external forces
- ◆ Collisions among discrete particles in the system, for example, are excluded
- ◆ However, if a particle interacts more strongly with the collective fields of the other particles than with its nearest neighbours, the Vlasov equation still applies if one treats the collective fields on the same footing as the external fields
- ◆ This in fact forms the basis of treating the collective instabilities using the Vlasov technique

## VLASOV FORMALISM (6/7)

- ◆ Vlasov equation for a system of particles subject to simple harmonic motions with Hamiltonian

$$H = \omega \frac{q^2 + p^2}{2}$$

- Equations of motion

$$\dot{q} = \frac{\partial H}{\partial p} = p \omega$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -q \omega$$

$$\Rightarrow \ddot{q} + \omega^2 q = 0$$

Motion of a harmonic oscillator

- Going to polar coordinates, the Vlasov equation writes

$$q = r \cos \phi$$

$$p = -r \sin \phi$$

$$\frac{\partial \Psi}{\partial t} + \dot{r} \frac{\partial \Psi}{\partial r} + \dot{\phi} \frac{\partial \Psi}{\partial \phi} = 0$$

## VLASOV FORMALISM (7/7)

- ◆ As  $r$  is a constant of motion  $\Rightarrow \dot{r} = 0$

$$\Rightarrow \frac{\partial \Psi}{\partial t} + \omega \frac{\partial \Psi}{\partial \phi} = 0$$

$$\phi = \omega t$$

$$\Rightarrow \frac{\partial \Psi}{\partial t} = -\omega \frac{\partial \Psi}{\partial \phi} = -\frac{\partial \Psi}{\partial t}$$

$$\Rightarrow \frac{\partial \Psi}{\partial t} = \frac{\partial \Psi}{\partial \phi} = 0$$

$$\Rightarrow \Psi \text{ depends only on } r: \Psi(r)$$

- ◆ Once the initial distribution of the beam is given at time  $t = 0$ , the distribution at time  $t$  is obtained by rigidly rotating the initial distribution in phase space angle  $\phi$  at a constant angular speed  $\omega$
- ◆ A stationary distribution is any function of  $r$ , or equivalently any function of the Hamiltonian  $H$

# HEAD-TAIL INSTABILITY (1/43)

## SINGLE-PARTICLE TRANSVERSE MOTIONS

- ◆ The transverse motions of a test particle in a bunch are described by six coordinates
- ◆ 2 of them are related to the longitudinal phase space => The parameters  $(\tau_i, \dot{\tau}_i)$  or  $(\hat{\tau}_i, \psi_i)$  will be used
- ◆ Here,  $\tau_i$  represents the time interval between the passage of the synchronous particle and the test particle
- ◆ A purely linear synchrotron oscillation around the synchronous particle at frequency  $\omega_s$  is assumed

$$\ddot{\tau}_i + \omega_s^2 \tau_i = 0$$

$$\tau_i = \hat{\tau}_i \cos(\omega_s t + \psi_i)$$

- ◆ The motion in the longitudinal plane is assumed to be stable (no coherent effect)

## HEAD-TAIL INSTABILITY (2/43)

- ◆ The other 4 parameters are 2 pairs of coordinates  $(x_i, \dot{x}_i)$  or  $(\hat{x}_i, \varphi_{x,i})$ , and  $(y_i, \dot{y}_i)$  or  $(\hat{y}_i, \varphi_{y,i})$
- ◆ They are related to the transverse phase spaces (horizontal and vertical respectively)
- ◆ Here,  $x_i$  and  $y_i$  are the betatron coordinates,  $\varphi_{x,i}$  and  $\varphi_{y,i}$  are the betatron phases at time  $t$ . The solution of the equation of unperturbed motion, e.g. in the horizontal plane, is written as

$$x_i = \hat{x}_i \cos(\varphi_{x,i})$$

- ◆ **Reminder: The horizontal betatron frequency is given by (see Coasting beams)**

$$\varphi_{x,i} = \omega_{x,i} t + \left( \omega_{\xi_x} - Q_{x0} \Omega_0 \right) \tau_i + \varphi_{0x,i}$$

$$\omega_{x,i} = Q_{x0} \Omega_0 + \dot{\varphi}_{x,i}(\hat{x}_i, \hat{y}_i)$$

$$\omega_{\xi_x} = Q_{x0} \Omega_0 \frac{\xi_x}{\eta}$$

Horizontal  
chromatic  
frequency

## HEAD-TAIL INSTABILITY (3/43)

=> In the absence of perturbation the horizontal coordinate satisfies

$$\ddot{x}_i - \frac{\ddot{\varphi}_{x,i}}{\dot{\varphi}_{x,i}} \dot{x}_i + \dot{\varphi}_{x,i}^2 x_i = 0$$

- ◆ In the presence of electromagnetic fields induced by the beam, the previous equation is modified to

$$\ddot{x}_i - \frac{\ddot{\varphi}_{x,i}}{\dot{\varphi}_{x,i}} \dot{x}_i + \dot{\varphi}_{x,i}^2 x_i = \frac{e}{m_0 \gamma} \left[ \vec{E} + \vec{v} \times \vec{B} \right]_x \left( t, \vartheta = \Omega_i(t - \tau_i) \right)$$

The electromagnetic fields must be expressed like this when following the particle along its trajectory

## HEAD-TAIL INSTABILITY (4/43)

### SINGLE-PARTICLE TRANSVERSE SIGNALS

- ◆ The horizontal signal  $s_{x,i}(t, \vartheta)$  induced at a perfect pick-up electrode (infinite bandwidth) at angular position  $\vartheta$  in the ring by the off-centered test particle is given by

$$s_{x,i}(t, \vartheta) = s_{z,i}(t, \vartheta) x_i(t) = s_{z,i}(t, \vartheta) \hat{x}_i \cos(\varphi_{x,i})$$

where  $s_{z,i}(t, \vartheta)$  is the current signal of the particle that moves in the external guide field (no self field added)

- ◆ At time  $t = 0$ , the synchronous particle starts from  $\vartheta = 0$  and reaches the pick-up electrode at times  $t_k^0$  satisfying

$$\Omega_0 t_k^0 = \vartheta + 2k\pi, \quad -\infty \leq k \leq +\infty$$

## HEAD-TAIL INSTABILITY (5/43)

- ◆ The test particle is delayed by  $\tau_i$ . It goes through the electrode at times  $t_k$  given by

$$t_k = t_k^0 + \tau_i$$

- ◆ The current signal induced by the test particle is a series of impulses delivered on each passage

$$s_{z,i}(t, \vartheta) = e \sum_{k=-\infty}^{k=+\infty} \delta\left(t - \tau_i - \frac{\vartheta}{\Omega_0} - \frac{2k\pi}{\Omega_0}\right)$$

- ◆ In the time domain,  $s_{x,i}(t, \vartheta)$  consists of a series of impulses, the amplitude of which  $x_i$  changes at each passage through the electrode

$$s_{x,i}(t, \vartheta) = e \hat{x}_i \cos(\varphi_{x,i}) \sum_{k=-\infty}^{k=+\infty} \delta\left(t - \tau_i - \frac{\vartheta}{\Omega_0} - \frac{2k\pi}{\Omega_0}\right)$$

## HEAD-TAIL INSTABILITY (6/43)

- ◆ Developing  $\cos(\varphi_{x,i})$  into exponential functions and using the following equation

$$\sum_{k=-\infty}^{k=+\infty} \delta\left(u - \frac{2k\pi}{\Omega_0}\right) = \frac{\Omega_0}{2\pi} \sum_{k=-\infty}^{k=+\infty} e^{jk\Omega_0 u}$$

$$\Rightarrow s_{x,i}(t, \vartheta) = \frac{e\Omega_0}{2\pi} \hat{x}_i \frac{e^{j\varphi_{x,i}} + e^{-j\varphi_{x,i}}}{2} \sum_{k=-\infty}^{k=+\infty} e^{jk[\Omega_0(t-\tau_i) - \vartheta]}$$

- ◆ Using now

$$e^{-ju\hat{\tau}_i} \cos(\omega_s t + \psi_i) = \sum_{m=-\infty}^{m=+\infty} j^{-m} J_m(u\hat{\tau}_i) e^{jm(\omega_s t + \psi_i)}$$

**Bessel function of mth order**

## HEAD-TAIL INSTABILITY (7/43)

$$\Rightarrow s_{x,i}(t, \vartheta) = \frac{e^{\Omega_0}}{4\pi} \hat{x}_i e^{j(\omega_{x,i} t + \varphi_{0x,i})} \sum_{m,k} j^{-m} J_{m,x}(k, \hat{\tau}_i) e^{j[(k\Omega_0 + m\omega_s)t + m\psi_i - k\vartheta]}$$

neglecting the complex conjugate  $e^{-j\varphi_{x,i}}$  and with

$$J_{m,x}(k, \hat{\tau}_i) = J_m \left\{ \left[ (k + Q_{x,i}) \Omega_0 - \omega_{\xi_x} \right] \hat{\tau}_i \right\} \approx J_m \left\{ \left[ (k + Q_{x0}) \Omega_0 - \omega_{\xi_x} \right] \hat{\tau}_i \right\}$$

◆ Using the Fourier transform, one obtains

$$s_{x,i}(\omega, \vartheta) = \frac{e^{\Omega_0}}{4\pi} \hat{x}_i e^{j\varphi_{0x,i}} \sum_{m,k} j^{-m} J_{m,x}(k, \hat{\tau}_i) \delta[\omega - (k\Omega_0 + \omega_{x,i} + m\omega_s)] e^{j(m\psi_i - k\vartheta)}$$

## HEAD-TAIL INSTABILITY (8/43)

=> The single particle spectrum is a line spectrum at frequencies

$$(k + Q_{x,i}) \Omega_0 + m \omega_s$$

- ◆ Around every betatron line  $(k + Q_{x,i}) \Omega_0$ , there is an infinite number of synchrotron satellites  $m$ , the amplitude of which is given by the Bessel function:

$$J_m \left\{ \left[ (k + Q_{x,i}) \Omega_0 - \omega_{\xi_x} \right] \hat{\tau}_i \right\}$$

- ◆ The important point here is that the spectrum is centered at the chromatic frequency

# HEAD-TAIL INSTABILITY (9/43)

## STATIONARY DISTRIBUTION

- ◆ In the absence of perturbation,  $\hat{x}_i$  ,  $\hat{y}_i$  and  $\hat{\tau}_i$  are constant during the motion
- ◆ Therefore, the stationary distribution is a function of the peak amplitudes only

$$\Psi_0(\hat{x}_i, \hat{y}_i, \hat{\tau}_i)$$

- ◆ No correlation between horizontal, vertical and longitudinal planes is assumed and the stationary part is thus written as the product of three stationary distributions, one for the longitudinal phase space and one for each transverse phase space

$$\Psi_0 = f_{x0}(\hat{x}_i) f_{y0}(\hat{y}_i) g_0(\hat{\tau}_i)$$

$$\int_{\hat{\tau}_i=0}^{\hat{\tau}_i=\tau_b/2} g_0(\hat{\tau}_i) \hat{\tau}_i d\hat{\tau}_i = \frac{1}{2\pi}$$

$$\int_{\hat{x}_i=0}^{\hat{x}_i=+\infty} f_{x0}(\hat{x}_i) \hat{x}_i d\hat{x}_i = \frac{1}{2\pi}$$

$$\int_{\hat{y}_i=0}^{\hat{y}_i=+\infty} f_{y0}(\hat{y}_i) \hat{y}_i d\hat{y}_i = \frac{1}{2\pi}$$

## HEAD-TAIL INSTABILITY (10/43)

- ◆ Since on average, the beam center of mass is on axis, e.g. the horizontal signal as well as the horizontal dipolar magnetic field induced by the stationary distribution are null

Number of particles in the bunch

$$S_x(t, \vartheta) = N_b \int \int \int \int \int \int s_{x,i}(t, \vartheta) f_{x0}(\hat{x}_i) f_{y0}(\hat{y}_i) g_0(\hat{\tau}_i) \hat{x}_i \hat{y}_i \hat{\tau}_i \times \\ d\hat{x}_i d\hat{y}_i d\hat{\tau}_i d\varphi_{0x,i} d\varphi_{0y,i} d\psi_i = 0$$

## HEAD-TAIL INSTABILITY (11/43)

### PERTURBATION

$$\Psi_x = \Psi_0 + \Delta\Psi_x$$

- ◆ In order to get some dipolar fields, density perturbations  $\Delta\Psi_x$  that describe beam center-of-mass displacements along the bunch are assumed
- ◆ The mathematical form of the perturbations is suggested by the single-particle signals
- ◆ The kind of perturbation we are looking for is the rigid-dipolar mode. This is the mode for which the stationary distribution  $f_{x0}(\hat{x}_i)$  is displaced from the origin by a small amount and rotate rigidly about the origin
- ◆ Expanding this distribution to 1<sup>st</sup> order and considering a single value of  $m$  (i.e. considering the case of low intensity coherent modes of oscillation, in which the betatron frequency shift remains small when compared to the incoherent synchrotron frequency  $\omega_s$  ), one then has for the amplitudes of the perturbations

## HEAD-TAIL INSTABILITY (12/43)

$$h_m^x(\hat{x}_i, \hat{y}_i, \hat{\tau}_i) = \frac{df_{x0}(\hat{x}_i)}{d\hat{x}_i} \hat{x}_m(\hat{\tau}_i) f_{y0}(\hat{y}_i) g_0(\hat{\tau}_i)$$

where  $\hat{x}_m(\hat{\tau}_i)$  is the coherent (average) horizontal peak betatron amplitude associated with a given synchrotron orbit  $\hat{\tau}_i$ . Furthermore, because of the integral over  $\varphi_{0x,i}$ ,  $\varphi_{0y,i}$  and  $\psi_i$ , the transverse signals induced would be null unless one introduces the complex conjugates of  $e^{j(\varphi_{0x,i} + m\psi_i)}$  in the perturbation term  $\Rightarrow$  The betatron phases and synchrotron phase are chosen in order to satisfy

$$\varphi_{0x,i} + m\psi_i = 0$$

$$\Rightarrow \Delta\Psi_x = h_m^x(\hat{x}_i, \hat{y}_i, \hat{\tau}_i) e^{j\Delta\omega_{c,m}^{x,i} t} \quad \text{with} \quad \Delta\omega_{c,m}^{x,i} = \omega_c - [\omega_{x,i}(\hat{x}_i, \hat{y}_i) + m\omega_s]$$

Coherent (angular) betatron frequency to be determined

## HEAD-TAIL INSTABILITY (13/43)

- ◆ In the time domain, the horizontal signal takes the form (for a single value  $m$ )

$$S_x(t, \vartheta) = \frac{e^{\Omega_0}}{4\pi} N_b \int \int \int \int \int \int \hat{x}_i \sum_k j^{-m} J_{m,x}(k, \hat{\tau}_i) \times \\ h_m^x(\hat{x}_i, \hat{y}_i, \hat{\tau}_i) e^{-jk\vartheta} e^{j(k\Omega_0 + \omega_c)t} \hat{x}_i \hat{y}_i \hat{\tau}_i d\hat{x}_i d\hat{y}_i d\hat{\tau}_i d\varphi_{0x,i} d\varphi_{0y,i} d\psi_i$$

- ◆ In the frequency domain, it becomes

$$S_x(\omega, \vartheta) = 2\pi^2 I_b \sum_k e^{-jk\vartheta} \sigma_{x,m}(k) \delta[\omega - (k\Omega_0 + \omega_c)]$$

with  $\sigma_{x,m}(k) = j^{-m} 2\pi \int \int \int h_m^x(\hat{x}_i, \hat{y}_i, \hat{\tau}_i) J_{m,x}(k, \hat{\tau}_i) \hat{x}_i^2 d\hat{x}_i \hat{y}_i d\hat{y}_i \hat{\tau}_i d\hat{\tau}_i$

$$I_b = N_b e f_0$$

**Bunch current**

## HEAD-TAIL INSTABILITY (14/43)

- ◆ In comparison with the rich spectrum of the test particle, a single synchrotron satellite remains
- ◆ The perturbation is coherent with respect to the satellite number  $m$
- ◆ By means of the perturbation, the transverse initial conditions of the particles in the bunch have been arranged. The result of this perturbation is that the position of the center of mass changes along the bunch. The horizontal phase space distribution rotates not at incoherent frequency  $Q_{x0} \Omega_0 + m \omega_s$  exactly but at frequency  $\omega_c$ , due to the perturbations (wake fields) and the frequency spread

# HEAD-TAIL INSTABILITY (15/43)

## TRANSVERSE COUPLING IMPEDANCE

- ◆ The coupling impedance  $Z_x$ , which gather all the characteristics of the electromagnetic response of a machine to a passing particle, allow us to express the transverse fields in terms of transverse signals

$$\left[ \vec{E} + \vec{v} \times \vec{B} \right]_{x,y} (t, \vartheta) = \frac{-j\beta}{2\pi R} \int Z_{x,y}(\omega) S_{x,y}(\omega, \vartheta) e^{j\omega t} d\omega$$

in  $\Omega / m$

in A m

# HEAD-TAIL INSTABILITY (16/43)

## EQUATION OF COHERENT MOTION

- ◆  $\Psi_x$  satisfies the “reduced” Vlasov equation  $\Psi_x = f_{y0}(\hat{y}_i) \int_{\hat{y}_i=0}^{\hat{y}_i=+\infty} \int_{\varphi_{0y,i}=0}^{\varphi_{0y,i}=2\pi} \Psi \hat{y}_i d\hat{y}_i d\varphi_{0y,i}$

$$\frac{\partial \Psi_x}{\partial t} + \frac{\partial \Psi_x}{\partial \hat{x}_i} \dot{\hat{x}}_i + \frac{\partial \Psi_x}{\partial \varphi_{0x,i}} \dot{\varphi}_{0x,i} + \frac{\partial \Psi_x}{\partial \hat{\tau}_i} \dot{\hat{\tau}}_i + \frac{\partial \Psi_x}{\partial \psi_i} \dot{\psi}_i = 0$$

- ◆ Dropping the 2<sup>nd</sup> order terms with respect to the perturbations, yields

$$j \Delta \omega_{c,m}^{x,i} h_m^x(\hat{x}_i, \hat{y}_i, \hat{\tau}_i) e^{j \Delta \omega_{c,m}^{x,i} t} = - \frac{df_{x0}(\hat{x}_i)}{d\hat{x}_i} \dot{\hat{x}}_i f_{y0}(\hat{y}_i) g_0(\hat{\tau}_i)$$

- ◆ The expression of  $\dot{\hat{x}}_i$  can be drawn from the single-particle horizontal equation of motion

## HEAD-TAIL INSTABILITY (17/43)

$$\dot{\hat{x}}_i = \frac{d}{dt}(\hat{x}_i) = \frac{d}{dt} \left[ x_i^2 + \left( \frac{\dot{x}_i}{\dot{\varphi}_{x,i}} \right)^2 \right]^{1/2} = -\frac{\sin(\varphi_{x,i})}{\dot{\varphi}_{x,i}} F_x \approx -\frac{j}{2\omega_{x0}} e^{-j\varphi_{x,i}} F_x$$

with

$$F_x = \frac{e}{m_0 \gamma} \left[ \vec{E} + \vec{v} \times \vec{B} \right]_x \left( t, \vartheta = \Omega_0 (t - \tau_i) \right)$$

- ◆ Here,  $\dot{\varphi}_{x,i}$  has been approximated by  $\omega_{x0}$  and  $\sin(\varphi_{x,i})$  by  $(j/2) e^{-j\varphi_{x,i}}$ , since the other component can be ignored if the frequency shift is small compared to the betatron frequency

- ◆ This leads to

$$-\frac{j}{2\omega_{x0}} e^{-j\varphi_{x,i}} F_x = -\frac{e \pi I_b}{2 m_0 c \gamma Q_{x0}} \sum_k Z_x(\omega_k^x) \sigma_{x,m}(k) j^m J_{m,x}(k, \hat{\tau}_i) e^{j \Delta \omega_{c,m}^{x,i} t}$$

## HEAD-TAIL INSTABILITY (18/43)

where  $\omega_k^x = (k + Q_{x0}) \Omega_0 + m\omega_s, \quad -\infty \leq k \leq +\infty$

$$\Rightarrow j\Delta\omega_{c,m}^{x,i} h_m^x(\hat{x}_i, \hat{y}_i, \hat{\tau}_i) e^{j\Delta\omega_{c,m}^{x,i} t} = \frac{df_{x0}(\hat{x}_i)}{d\hat{x}_i} f_{y0}(\hat{y}_i) g_0(\hat{\tau}_i) \times$$

$$\frac{e\pi I_b}{2m_0 c \gamma Q_{x0}} \sum_k Z_x(\omega_k^x) \sigma_{x,m}(k) j^m J_{m,x}(k, \hat{\tau}_i) e^{j\Delta\omega_{c,m}^{x,i} t}$$

- ◆ Multiplying both sides by  $e^{j\varphi_{x,i}} e^{-jp\Omega_0\tau_i}$ , where  $p$  is an integer, developing it in Bessel functions (as seen before) and retaining only one value  $m$ , one obtains

$$j\Delta\omega_{c,m}^{x,i} h_m^x(\hat{x}_i, \hat{y}_i, \hat{\tau}_i) J_{m,x}(p, \hat{\tau}_i) = \frac{df_{x0}(\hat{x}_i)}{d\hat{x}_i} f_{y0}(\hat{y}_i) g_0(\hat{\tau}_i) \times$$

$$\frac{e\pi I_b}{2m_0 c \gamma Q_{x0}} \sum_k Z_x(\omega_k^x) \sigma_{x,m}(k) j^m J_{m,x}(k, \hat{\tau}_i) J_{m,x}(p, \hat{\tau}_i)$$

## HEAD-TAIL INSTABILITY (19/43)

- Using the definition of  $h_m^x(\hat{x}_i, \hat{y}_i, \hat{\tau}_i)$ , multiplying both sides by  $\hat{x}_i^2 \hat{y}_i$ , integrating over  $\hat{x}_i$  and  $\hat{y}_i$  values and using

$$\int_{\hat{x}_i=0}^{\hat{x}_i=+\infty} \frac{df_{x0}(\hat{x}_i)}{d\hat{x}_i} \hat{x}_i^2 d\hat{x}_i = -2 \int_{\hat{x}_i=0}^{\hat{x}_i=+\infty} f_{x0}(\hat{x}_i) \hat{x}_i d\hat{x}_i = -\frac{1}{\pi}$$

one obtains

$$I_x^{-1} \hat{x}_m(\hat{\tau}_i) J_{m,x}(p, \hat{\tau}_i) g_0(\hat{\tau}_i) = -\frac{j e \pi I_b}{2 m_0 c \gamma Q_{x0}} \sum_k Z_x(\omega_k^x) \sigma_{x,m}(k) j^m J_{m,x}(k, \hat{\tau}_i) J_{m,x}(p, \hat{\tau}_i) g_0(\hat{\tau}_i)$$

with

$$\begin{aligned} \sigma_{x,m}(k) &= j^{-m} 2\pi \int \int \int \frac{df_{x0}(\hat{x}_i)}{d\hat{x}_i} \hat{x}_m(\hat{\tau}_i) J_{m,x}(k, \hat{\tau}_i) f_{y0}(\hat{y}_i) g_0(\hat{\tau}_i) \hat{x}_i^2 d\hat{x}_i \hat{y}_i d\hat{y}_i \hat{\tau}_i d\hat{\tau}_i \\ &= -\frac{j^{-m}}{\pi} \int \hat{x}_m(\hat{\tau}_i) J_{m,x}(k, \hat{\tau}_i) g_0(\hat{\tau}_i) \hat{\tau}_i d\hat{\tau}_i \end{aligned}$$

## HEAD-TAIL INSTABILITY (20/43)

and

$$I_x = \int_{\hat{x}_i=0}^{\hat{x}_i=+\infty} \int_{\hat{y}_i=0}^{\hat{y}_i=+\infty} \frac{-2\pi^2 \frac{df_{x0}(\hat{x}_i)}{d\hat{x}_i} \hat{x}_i^2 f_{y0}(\hat{y}_i) \hat{y}_i}{\omega_c - \omega_{x,i}(\hat{x}_i, \hat{y}_i) - m\omega_s} d\hat{x}_i d\hat{y}_i$$

- ◆ Multiplying both sides by  $\hat{\tau}_i$  and summing over  $\hat{\tau}_i$  values, yields

$$I_x^{-1} \sigma_{x,m}(p) = \sum_k K_{x,m}^{pk} \sigma_{x,m}(k)$$

with

$$K_{x,m}^{pk} = \frac{jeI_b}{2m_0c\gamma Q_{x0}} Z_x(\omega_k^x) \int_{\hat{\tau}_i=0}^{\hat{\tau}_i=\tau_b/2} J_{m,x}(k, \hat{\tau}_i) J_{m,x}(p, \hat{\tau}_i) g_0(\hat{\tau}_i) \hat{\tau}_i d\hat{\tau}_i$$

## HEAD-TAIL INSTABILITY (21/43)

- ◆ In the absence of frequency spread, the previous equation can be written as an eigensystem

$$\det \left[ K_{x,m} - (\omega_c - \omega_{x0} - m\omega_s) I \right] = 0$$

where  $I$  is the identity matrix,  $K_{x,m}$  is the matrix whose elements are given by  $K_{x,m}^{pk}$

## HEAD-TAIL INSTABILITY (22/43)

- ◆ The procedure to obtain first order exact solutions, with realistic modes and a general interaction, thus consists of finding the eigenvalues and eigenvectors of the infinite complex matrix  $K_{x,m}$ . The result is an infinite number of modes  $mq$  ( $-\infty < q < +\infty$ ) of oscillation. To each mode, one can associate a coherent frequency shift  $(\omega_c - \omega_{x0} - m\omega_s)_q$  (qth eigenvalue of the matrix), a coherent spectrum  $\sigma_{x,mq}(k)$  (qth eigenvector of the matrix) and a coherent peak betatron amplitude distribution  $\hat{x}_{mq}(\hat{\tau}_i)$
- ◆ For numerical reasons, the matrix needs to be truncated, and thus only a finite frequency domain is explored
- ◆ Low order eigenvalues and eigenvectors of the matrix  $K_{x,m}$  can be found quickly by computation, using the relations

## HEAD-TAIL INSTABILITY (23/43)

$$\int_0^X J_m^2(ax) x dx = \frac{X^2}{2} [J'_m(aX)]^2 + \frac{1}{2} \left[ X^2 - \frac{m^2}{a^2} \right] J_m^2(aX)$$

$$\int_0^X x J_m(ax) J_m(bx) dx = \frac{X}{a^2 - b^2} [a J_m(bX) J_{m+1}(aX) - b J_m(aX) J_{m+1}(bX)]$$

for  $a^2 \neq b^2$

$$g_0(\hat{\tau}_i) = 4 / (\pi \tau_b^2)$$

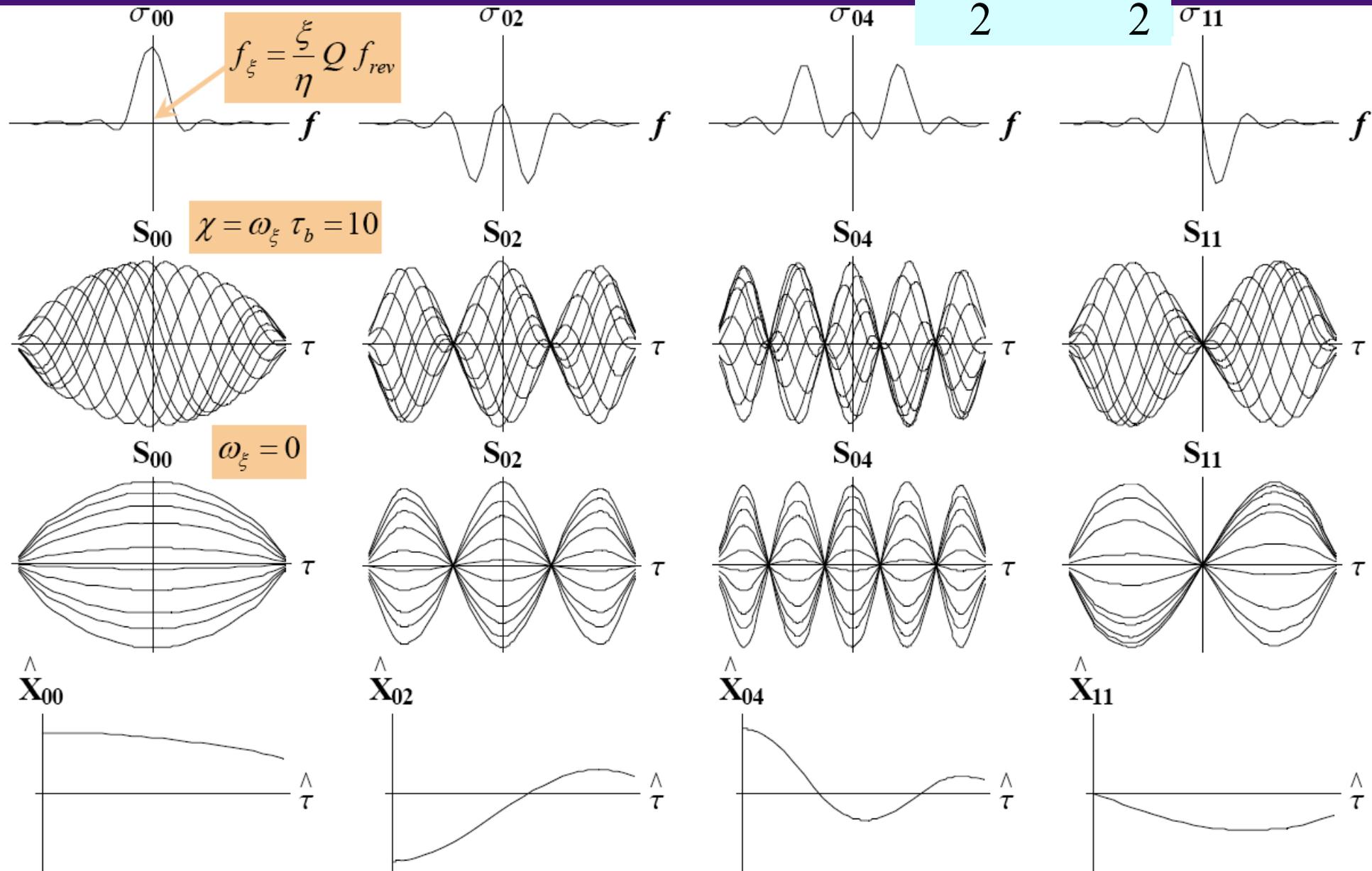
- ◆ The horizontal coherent oscillations (over several turns) of a “water-bag” bunch interacting with a constant inductive impedance are shown in the next slides for the first head-tail modes (solving the eigensystem)
- ◆ It is found that the spectrum of mode  $m q$  is peaked near  $\omega \approx (|q| + 1) \pi / \tau_b$  and extends over  $\sim \pm 2\pi / \tau_b$  (radians/second). The largest eigenvalue  $(\omega_c - \omega_{x0} - m\omega_s)_q$  takes the subscript  $q = m$ . Usually, only diagonal modes  $m m$  are referred to ( $m = 0$  dipolar mode,  $m = 1$  quadrupolar mode,...)

# HEAD-TAIL INSTABILITY (24/43)

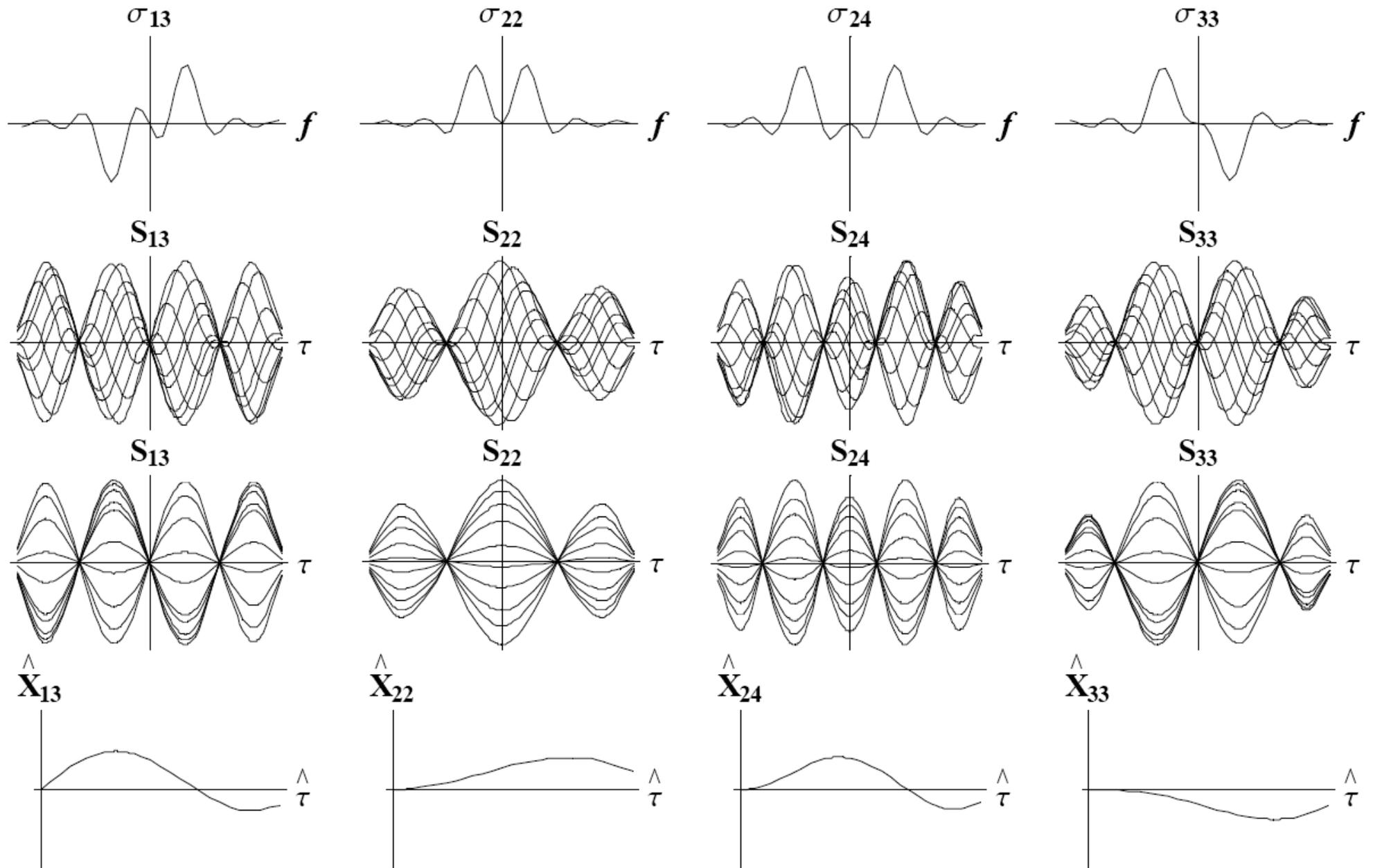
$$0 \leq \hat{\tau} \leq \tau_b / 2$$

$$-\frac{\tau_b}{2} \leq \tau \leq \frac{\tau_b}{2}$$

$$Q = x.13$$



# HEAD-TAIL INSTABILITY (25/43)



## HEAD-TAIL INSTABILITY (26/43)

- ◆ Finding the eigenvalues and eigenvectors of a complex matrix by computer can be difficult in some cases, and a simple approximate formula for the eigenvalues  $m m$  (which will be simply written  $m$  in the following) is useful in practice to have a rough estimate
- ◆ Multiplying both sides of

$$I_x^{-1} \hat{x}_m(\hat{\tau}_i) J_{m,x}(p, \hat{\tau}_i) g_0(\hat{\tau}_i) = -\frac{j e \pi I_b}{2 m_0 c \gamma Q_{x0}} \sum_k Z_x(\omega_k^x) \sigma_{x,m}(k) j^m J_{m,x}(k, \hat{\tau}_i) J_{m,x}(p, \hat{\tau}_i) g_0(\hat{\tau}_i)$$

(without frequency spread) by  $\hat{x}_m(\hat{\tau}_i) g_0(\hat{\tau}_i)$  and integrating over  $\hat{\tau}_i$  values, yields

$$(\omega_c - \omega_{x0} - m \omega_s) \int_{\hat{\tau}_i=0}^{\hat{\tau}_i=\tau_b/2} \hat{x}_m^2(\hat{\tau}_i) g_0(\hat{\tau}_i) \hat{\tau}_i d\hat{\tau}_i = \frac{j e \pi^2 I_b j^{2m}}{2 m_0 c \gamma Q_{x0}} \sum_{k=-\infty}^{k=+\infty} Z_x(\omega_k^x) \sigma_{x,m}^2(k)$$

# HEAD-TAIL INSTABILITY (27/43)

◆ Let's now show that

$$\pi^{-2} j^{-2m} \int_{\hat{\tau}_i=0}^{\hat{\tau}_i=\tau_b/2} \hat{x}_m^2(\hat{\tau}_i) g_0^2(\hat{\tau}_i) \hat{\tau}_i d\hat{\tau}_i = \frac{\Omega_0}{2} \sum_{k=-\infty}^{k=+\infty} \left| (k + Q_{x0}) \Omega_0 - \omega_{\xi_x} \right| \sigma_{x,m}^2(k)$$

From  $\sigma_{x,m}(k) = -\frac{j^{-m}}{\pi} \int \hat{x}_m(\hat{\tau}_i) J_{m,x}(k, \hat{\tau}_i) g_0(\hat{\tau}_i) \hat{\tau}_i d\hat{\tau}_i$ , one has

$$\frac{1}{2} \int_{\omega=-\infty}^{\omega=+\infty} |\omega| \sigma_{x,m}^2(\omega + \omega_{\xi_x} + m\omega_s) d\omega = \frac{\pi^{-2} j^{-2m}}{2} \int_{\hat{\tau}_i=0}^{\hat{\tau}_i=\tau_b/2} \int_{\hat{\tau}'_i=0}^{\hat{\tau}'_i=\tau_b/2} \left[ \int_{\omega=-\infty}^{\omega=+\infty} |\omega| J_m(\omega \hat{\tau}_i) J_m(\omega \hat{\tau}'_i) d\omega \right] \times \hat{x}_m(\hat{\tau}_i) g_0(\hat{\tau}_i) \hat{\tau}_i d\hat{\tau}_i \hat{x}_m(\hat{\tau}'_i) g_0(\hat{\tau}'_i) \hat{\tau}'_i d\hat{\tau}'_i$$

Using then the relation

$$\int_{\omega=-\infty}^{\omega=+\infty} |\omega| J_m(\omega \hat{\tau}_i) J_m(\omega \hat{\tau}'_i) d\omega = \frac{2}{\hat{\tau}_i} \delta(\hat{\tau}_i - \hat{\tau}'_i)$$

# HEAD-TAIL INSTABILITY (28/43)

one obtains

$$\begin{aligned} & \frac{1}{2} \int_{\omega=-\infty}^{\omega=+\infty} |\omega| \sigma_{x,m}^2 \left( \omega + \omega_{\xi_x} + m \omega_s \right) d\omega \\ &= \pi^{-2} j^{-2m} \int_{\hat{\tau}_i=0}^{\hat{\tau}_i=\tau_b/2} \int_{\hat{\tau}'_i=0}^{\hat{\tau}'_i=\tau_b/2} \hat{x}_m(\hat{\tau}_i) g_0(\hat{\tau}_i) d\hat{\tau}_i \hat{x}_m(\hat{\tau}'_i) g_0(\hat{\tau}'_i) \hat{\tau}'_i d\hat{\tau}'_i \delta(\hat{\tau}_i - \hat{\tau}'_i) \end{aligned}$$

Integrating over  $\hat{\tau}'_i$  values, yields

$$\frac{1}{2} \int_{\omega=-\infty}^{\omega=+\infty} |\omega| \sigma_{x,m}^2 \left( \omega + \omega_{\xi_x} + m \omega_s \right) d\omega = \pi^{-2} j^{-2m} \int_{\hat{\tau}_i=0}^{\hat{\tau}_i=\tau_b/2} \hat{x}_m^2(\hat{\tau}_i) g_0^2(\hat{\tau}_i) \hat{\tau}_i d\hat{\tau}_i$$

## HEAD-TAIL INSTABILITY (29/43)

**Sampling the function**  $|\omega| \sigma_{x,m}^2(\omega + \omega_{\xi_x} + m\omega_s)$  **at frequencies**  
 $\omega = (k + Q_{x0}) \Omega_0 - \omega_{\xi_x}$ , **one has**

$$\left[ |\omega| \sigma_{x,m}^2(\omega + \omega_{\xi_x} + m\omega_s) \right]_{\text{sampled}} = \Omega_0 \sum_{k=-\infty}^{k=+\infty} |\omega| \sigma_{x,m}^2(\omega + \omega_{\xi_x} + m\omega_s) \times \\ \delta\left[ \omega - (k + Q_{x0}) \Omega_0 + \omega_{\xi_x} \right]$$

**Then**

$$\frac{1}{2} \int_{\omega=-\infty}^{\omega=+\infty} \left[ |\omega| \sigma_{x,m}^2(\omega + \omega_{\xi_x} + m\omega_s) \right]_{\text{sampled}} d\omega \\ = \frac{\Omega_0}{2} \sum_{k=-\infty}^{k=+\infty} \left| (k + Q_{x0}) \Omega_0 - \omega_{\xi_x} \right| \sigma_{x,m}^2 \left[ (k + Q_{x0}) \Omega_0 + m\omega_s \right]$$

# HEAD-TAIL INSTABILITY (30/43)

## Using the approximation

$$\int_{\omega=-\infty}^{\omega=+\infty} \left[ |\omega| \sigma_{x,m}^2 \left( \omega + \omega_{\xi_x} + m\omega_s \right) \right]_{\text{simplified}} d\omega \approx \int_{\omega=-\infty}^{\omega=+\infty} |\omega| \sigma_{x,m}^2 \left( \omega + \omega_{\xi_x} + m\omega_s \right) d\omega$$

the initial equation is finally found

◆ Therefore, this leads to

Assuming a “water-bag bunch”

$$g_0(\hat{t}_i) = 4 / (\pi \tau_b^2)$$

$$\begin{aligned} & \left( \omega_c - \omega_{x0} - m\omega_s \right) \frac{\pi \tau_b^2 \Omega_0}{8} \sum_{k=-\infty}^{k=+\infty} \left| (k + Q_{x0}) \Omega_0 - \omega_{\xi_x} \right| \sigma_{x,m}^2(k) \\ &= \frac{jeI_b}{2m_0 c \gamma Q_{x0}} \sum_{k=-\infty}^{k=+\infty} Z_x(\omega_k^x) \sigma_{x,m}^2(k) \end{aligned}$$

## HEAD-TAIL INSTABILITY (31/43)

Furthermore, the mode  $m$  is peaked near the frequency  $(|m| + 1) \pi / \tau_b$  and the approximation  $(k + Q_{x0}) \Omega_0 - \omega_{\xi_x} \approx (|m| + 1) \pi / \tau_b$  can be used, which yields

$$(\omega_c - \omega_{x0} - m \omega_s) = (|m| + 1)^{-1} \frac{j e \beta I_b}{2 m_0 \gamma Q_{x0} \Omega_0 L_b}$$

$$\frac{\sum_{k=-\infty}^{k=+\infty} Z_x(\omega_k^x) \sigma_{x,m}^2(k)}{\sum_{k=-\infty}^{k=+\infty} \sigma_{x,m}^2(k)}$$

Full bunch length (in meters)

Effective impedance

## HEAD-TAIL INSTABILITY (32/43)

- ◆ The spectrum  $\sigma_{x,m}(k)$  depends on the interaction  $Z_x(\omega_k^x)$ . However, for a non exact but realistic set of modes, the previous equation with  $\sigma_{x,m}(k)$  given by the previous figure, can be used to find approximate eigenvalues for any  $Z_x(\omega_k^x)$
- ◆ A good (proportional) fitting of the power spectrum  $|\sigma_{x,m}(k)|^2$  of the previous figure is obtained by the following function

$$h_{m,m}(\omega) = \frac{\tau_b^2}{2\pi^4} (|m| + 1)^2 \frac{1 + (-1)^{|m|} \cos(\omega \tau_b)}{\left[ (\omega \tau_b / \pi)^2 - (|m| + 1)^2 \right]^2}$$

- ◆ The power spectrum of mode  $|m|$  is peaked near  $\omega \approx (|m| + 1) \pi / \tau_b$  and extends  $\sim \pm 2\pi / \tau_b$  (radians/second) as the discrete spectrum found numerically in the figure

# HEAD-TAIL INSTABILITY (33/43)

- ◆ Using the fitting function, Sacherer's formula for the transverse coherent frequency shifts of bunched beam modes is obtained

$$\Delta\omega_{c,mm}^x = (\omega_c - \omega_{x0} - m\omega_s) = (|m| + 1)^{-1} \frac{j e \beta I_b}{2 m_0 \gamma Q_{x0} \Omega_0 L_b} \frac{\sum_{k=-\infty}^{k=+\infty} Z_x(\omega_k^x) h_{m,m}(\omega_k^x - \omega_{\xi_x}^x)}{\sum_{k=-\infty}^{k=+\infty} h_{m,m}(\omega_k^x - \omega_{\xi_x}^x)}$$

with  $\omega_k^x = (k + Q_{x0}) \Omega_0 + m\omega_s$ ,  $-\infty \leq k \leq +\infty$ , for 1 bunch

and  $\omega_k^x = (k + Q_{x0}) \Omega_0 + m\omega_s$ ,  $k = n_x + k' M$ ,  $-\infty \leq k' \leq +\infty$

for  $M$  equi-spaced equi-populated bunches

Phase shift between 2 successive bunches  $2\pi n_{x,y} / M$

Coupled-bunch mode number

## HEAD-TAIL INSTABILITY (34/43)

- ◆ The function  $h_{m,m}(\omega - \omega_{\xi_x})$  is given by  $h_{m,m}(\omega - \omega_{\xi_x}) = \left| p_m(\omega - \omega_{\xi_x}) \right|^2$

where  $p_m(\omega - \omega_{\xi_x})$  is the Fourier transform of the signal  $p_m(t) e^{j\omega_{\xi_x} t}$ .  
 Here  $p_m(t)$  corresponds to sinusoidal modes given by

$$p_m(t) = \begin{cases} \cos\left[ (|m| + 1) \pi t / \tau_b \right], & m \text{ even} \\ \sin\left[ (|m| + 1) \pi t / \tau_b \right], & m \text{ odd} \end{cases}$$

- ◆ The difference signal from a beam position monitor has thus the form

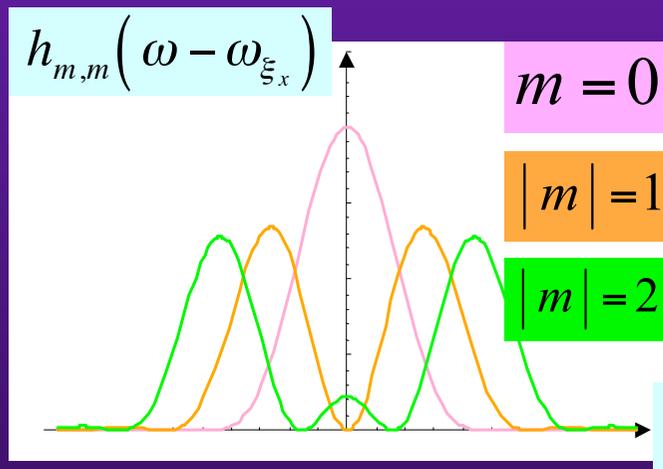
$$\Delta - \text{signal} \propto p_m(t) e^{j(\chi_x t / \tau_b + 2\pi k Q_{x0})}$$

For the kth revolution

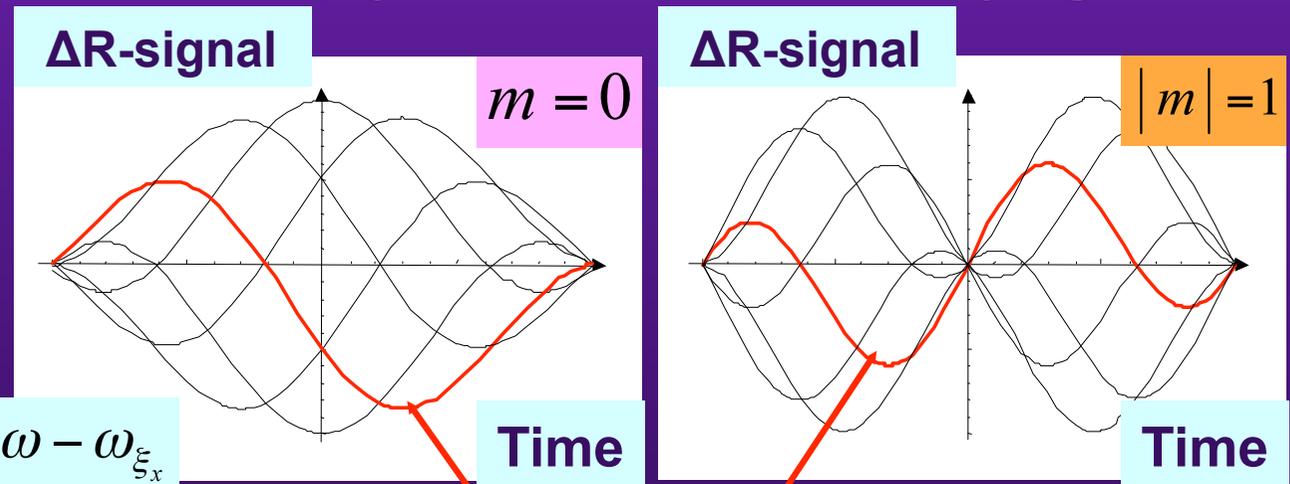
# HEAD-TAIL INSTABILITY (35/43)

- ◆ where  $\chi_x = \omega_{\xi_x} \tau_b$  (radians) is the total phase shift between head and tail
- ◆ The frequency of the wiggles along the bunch is determined by the horizontal chromatic frequency  $\omega_{\xi_x}$
- ◆ The number of nodes on separate superimposed revolutions gives the modulus of the head-tail mode number  $|m|$

Power spectrum



Pick-up (Beam Position Monitor) signal



One particular turn

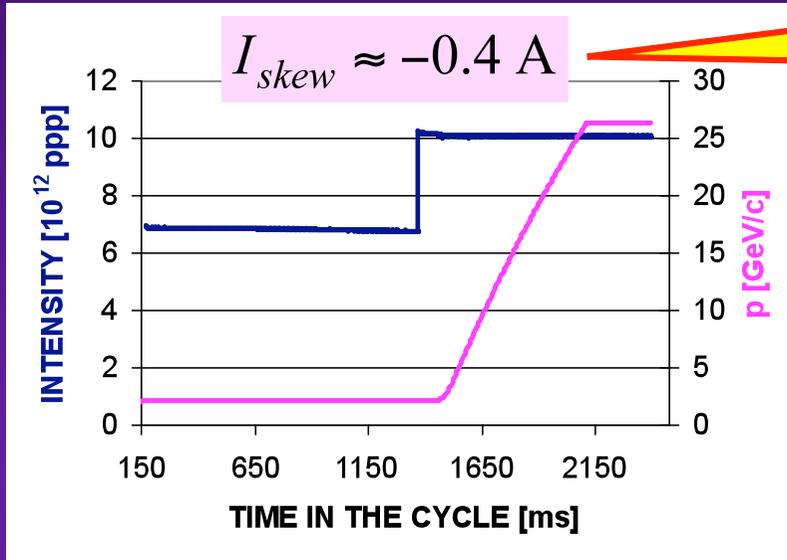
# HEAD-TAIL INSTABILITY (36/43)

## ◆ Example of slow Head-Tail single-bunch instability in the CERN PS

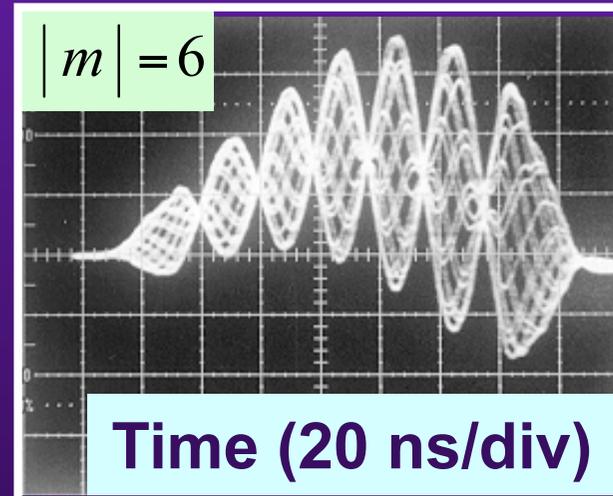
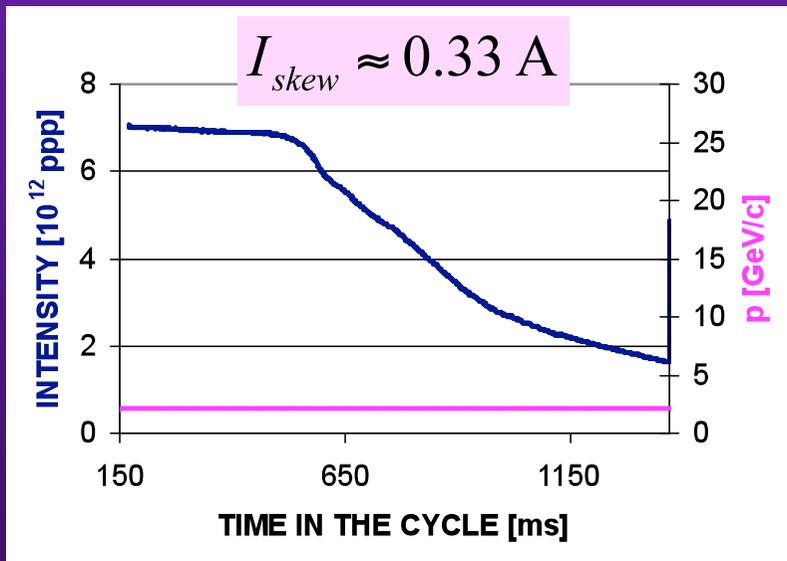
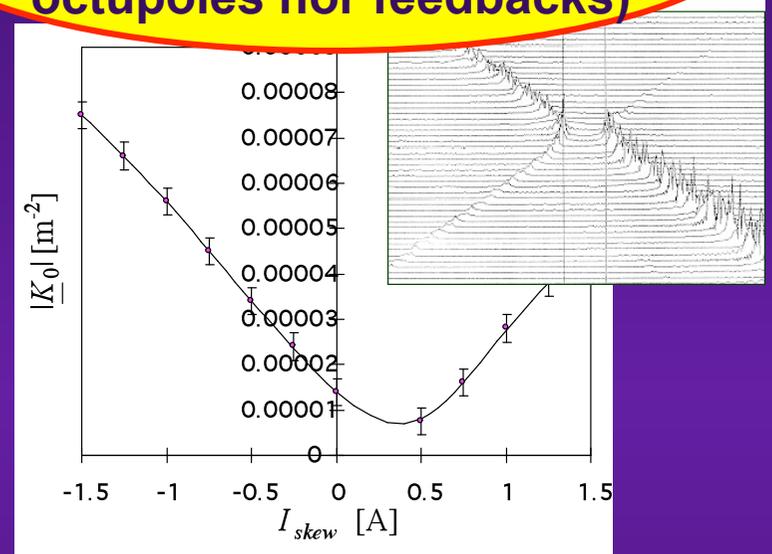
### ■ Measurements

$$Q_x = 6.22$$

$$Q_y = 6.25$$



Stabilisation by linear coupling only (i.e. with neither octupoles nor feedbacks)



## HEAD-TAIL INSTABILITY (37/43)

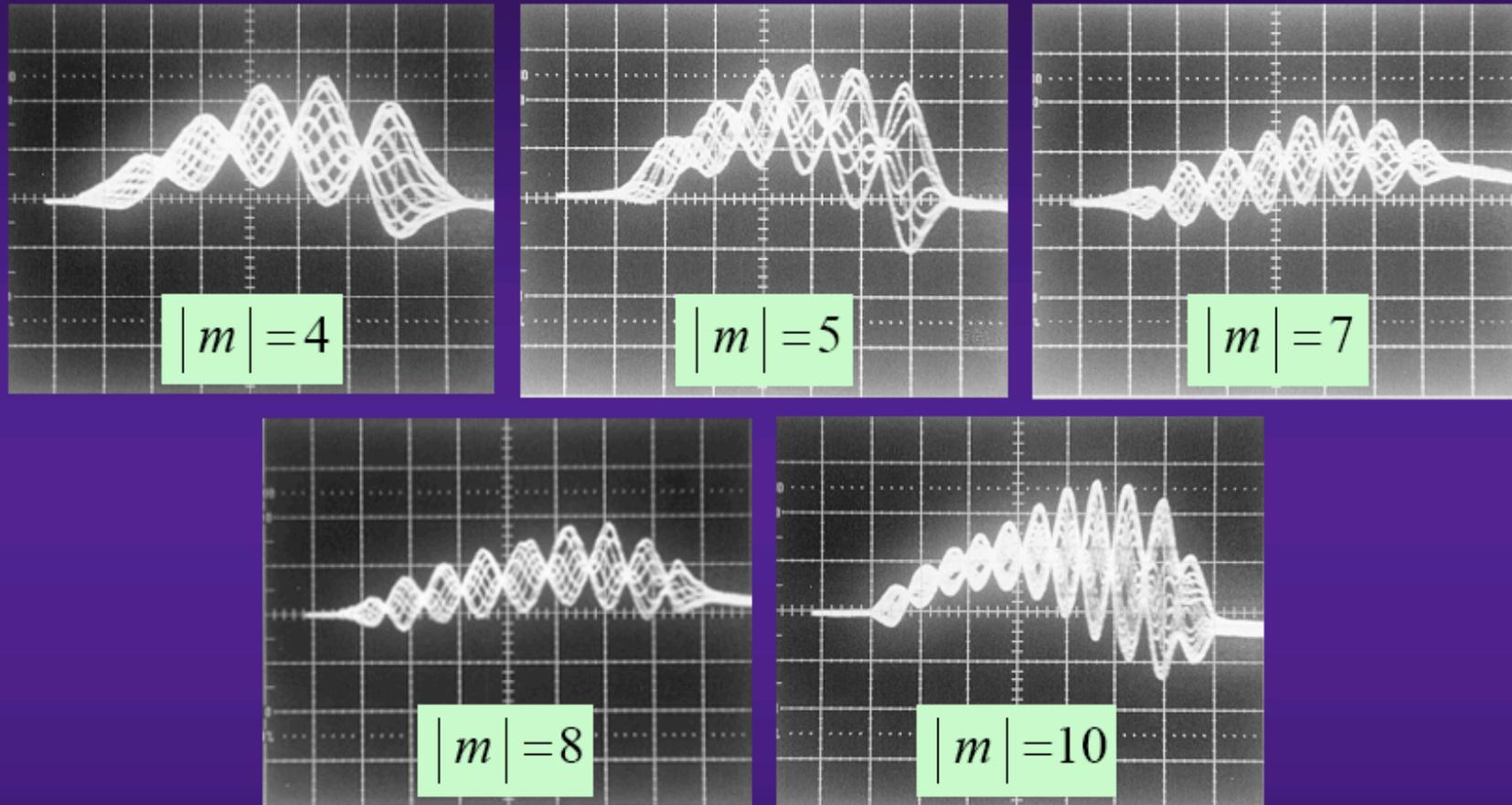
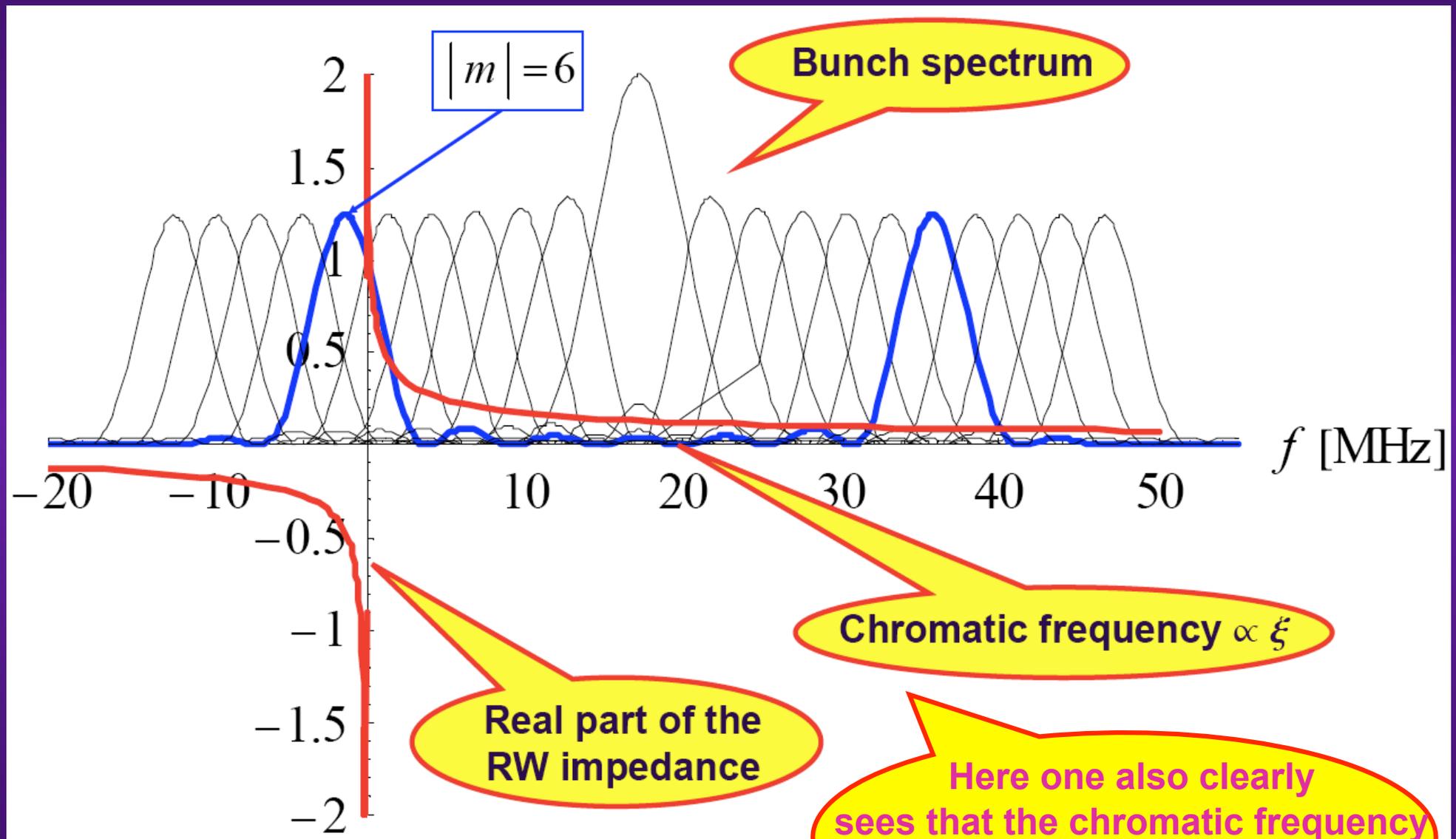


Figure 4: Measured  $\Delta R$  signals from a radial beam-position monitor during 20 consecutive turns, in the PS with minimum coupling [5]: (a)  $\xi_x \approx -0.5$ , (b)  $\xi_x \approx -0.7$ , (c)  $\xi_x \approx -1.1$ , (d)  $\xi_x \approx -1.2$ , (e)  $\xi_x \approx -1.3$ . Time scale: 20 ns/div.

# HEAD-TAIL INSTABILITY (38/43)

- Theoretical predictions



## HEAD-TAIL INSTABILITY (39/43)

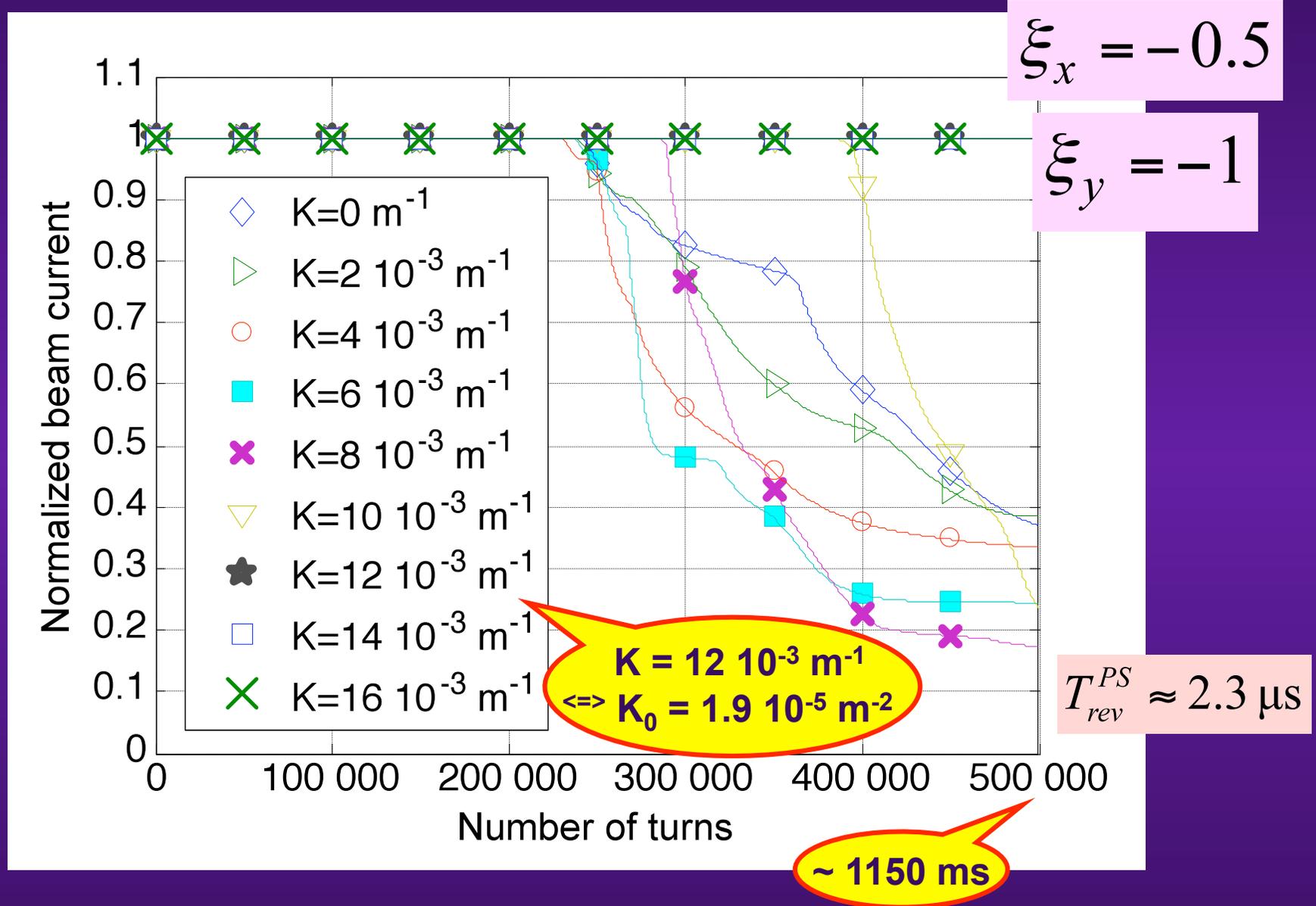
- ◆ **The results obtained with linear coupling between the transverse planes for coasting beams can also be extended to the case of bunched beams, using “equivalent dispersion relation coefficients”,**

$$U_{\text{eqx},y}^m = \text{Re} \left( \Delta\omega_{c,mm}^{x,y} \right)$$

$$V_{\text{eqx},y}^m = -\text{Im} \left( \Delta\omega_{c,mm}^{x,y} \right)$$

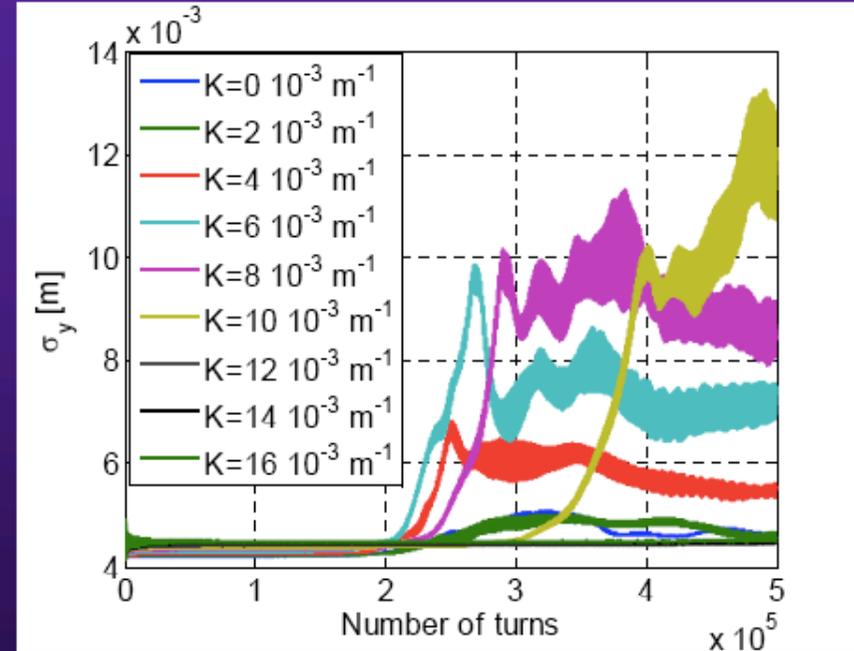
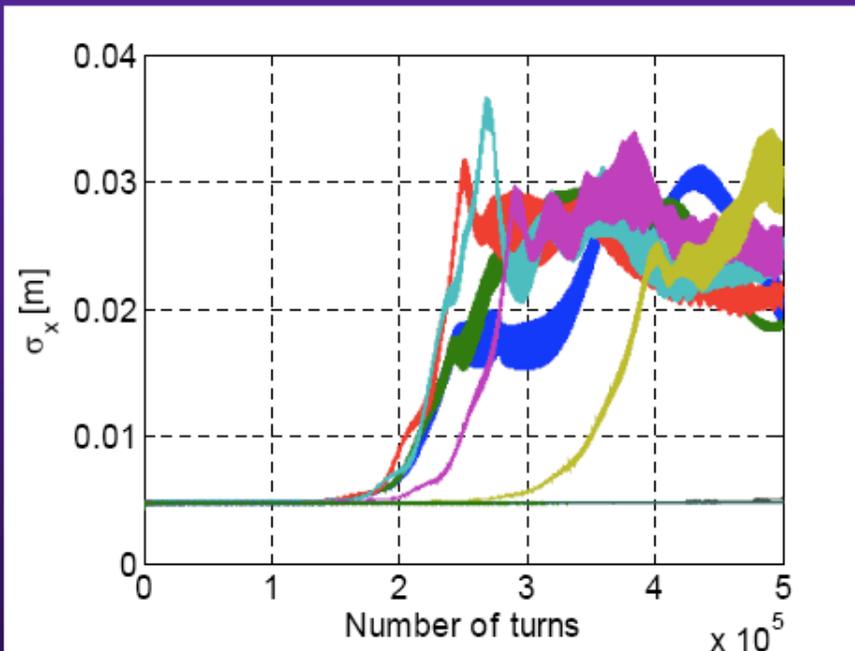
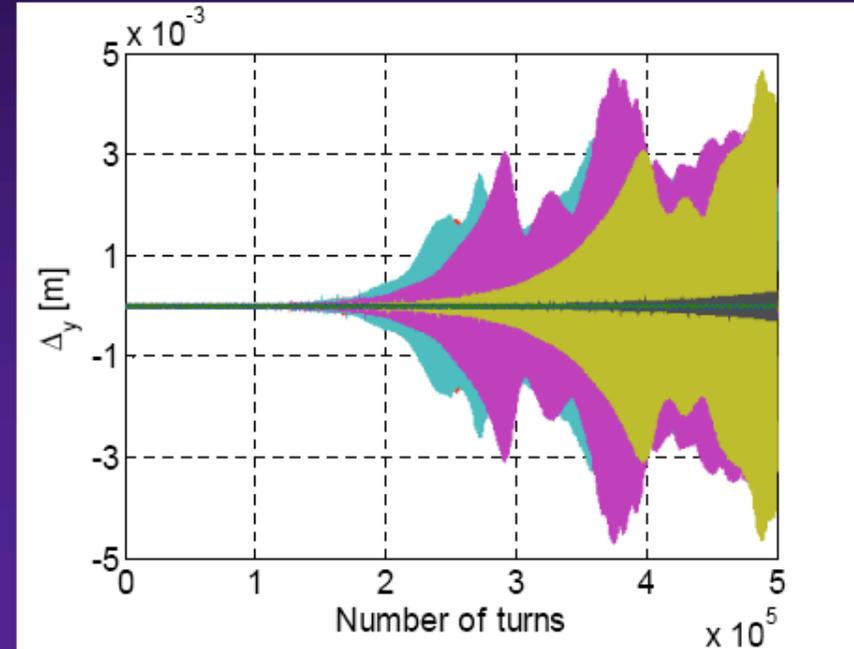
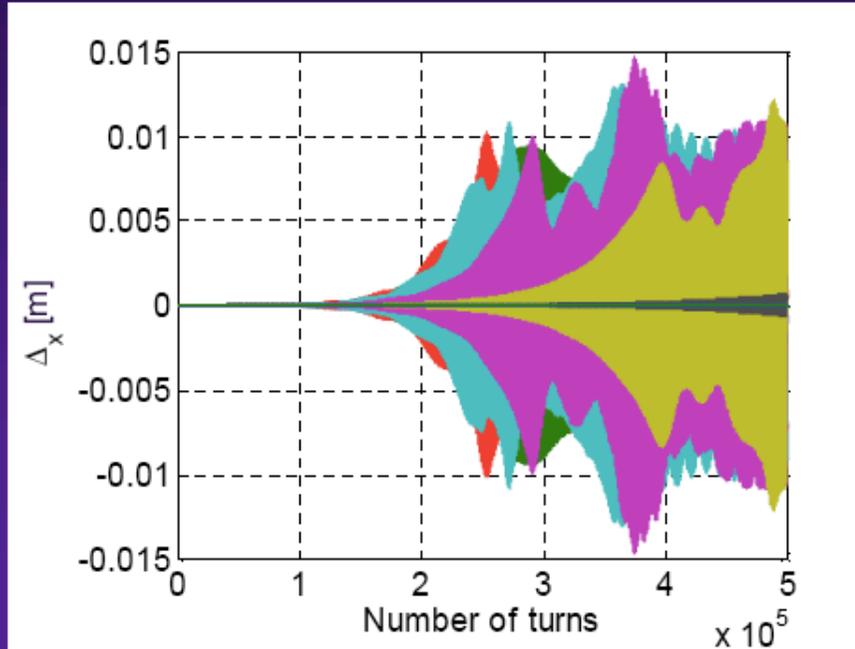
- ◆ **The beneficial effect of linear coupling has been checked with the HEADTAIL code assuming a round chamber (same impedance and betatron function in both planes) and only a different chromaticity in both planes (in the absence of frequency spread and SC etc.), to reveal the sharing of the instability growth rates (i.e. in fact of the chromaticities)**

# HEAD-TAIL INSTABILITY (40/43)



Courtesy of B. Salvant

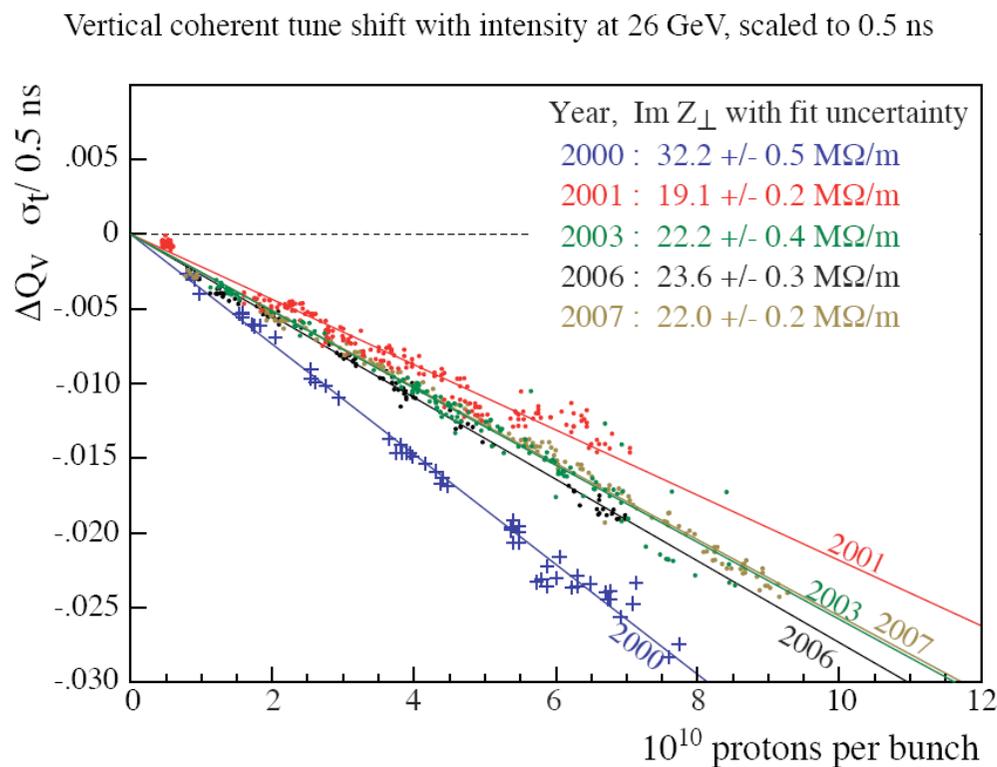
# HEAD-TAIL INSTABILITY (41/43)



Courtesy of B. Salvant

## HEAD-TAIL INSTABILITY (42/43)

- ◆ Finally, Sacherer's formula is also used to have: (1) an “estimate” of the imaginary part of the (effective) coupling impedance by measuring the coherent tune shift vs. intensity. However, one has to be careful here, remembering that in asymmetric structures, the quadrupolar term is also important



Same analysis and very similar beam parameters (~ 0.5 - 0.6 ns rms bunch length)  
The measured slopes can directly be compared. Estimated uncertainty ~ 10 - 20 %.

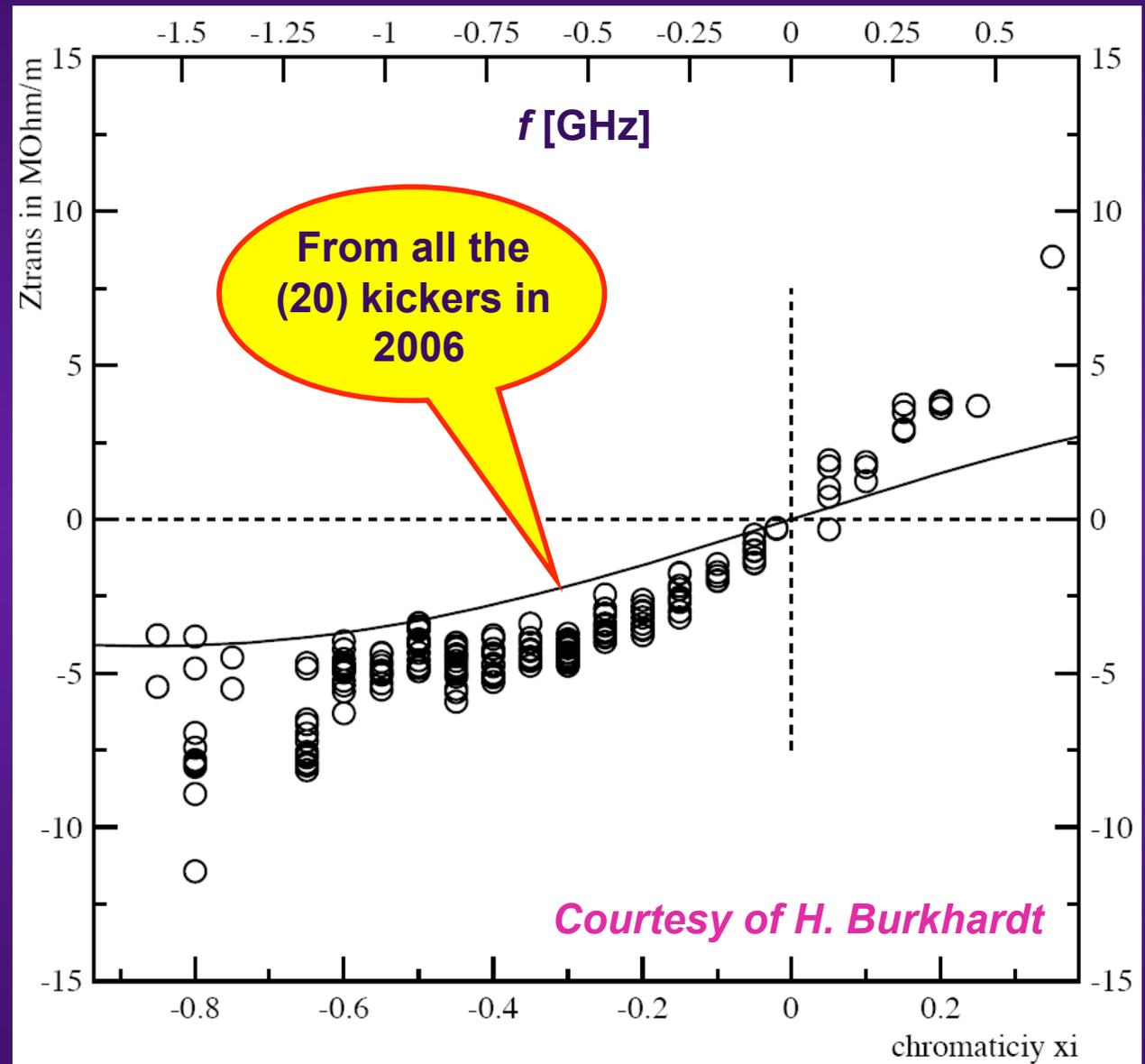
**CERN SPS at injection**

*Courtesy of H. Burkhardt*

# HEAD-TAIL INSTABILITY (43/43)

- ◆ (2) an “estimate” of the (effective) real part of the coupling impedance by measuring the head-tail growth/decay rates vs. chromaticity

CERN SPS at injection



## TMCI (1/40)

- ◆ If the intensity is too high, the different head-tail modes (which are standing-wave patterns) cannot be treated independently
- ◆ **Reminder: For 0 chromaticity, there is no Head-Tail instability**
- ◆ **However, even for 0 chromaticity, above a certain intensity threshold (called the Transverse Mode Coupling Instability threshold), 2 modes can couple leading to the Transverse Mode-Coupling Instability. In this case a traveling-wave pattern is propagating along the bunch**

## TMCI (2/40)

- ◆ Following the same formalism as before, the starting point is a formula we derived before

$$I_x^{-1} \hat{x}_m(\hat{\tau}_i) J_{m,x}(p, \hat{\tau}_i) g_0(\hat{\tau}_i) = -\frac{j e \pi I_b}{2 m_0 c \gamma Q_{x0}} \sum_k Z_x(\omega_k^x) \sigma_{x,m}(k) j^m J_{m,x}(k, \hat{\tau}_i) J_{m,x}(p, \hat{\tau}_i) g_0(\hat{\tau}_i)$$

which, without frequency spread is written

$$\begin{aligned} & \left[ \omega_c - \omega_{x0} - m \omega_s \right] \hat{x}_m(\hat{\tau}_i) J_{m,x}(p, \hat{\tau}_i) g_0(\hat{\tau}_i) \\ &= -\frac{j e \pi I_b}{2 m_0 c \gamma Q_{x0}} \sum_k Z_x(\omega_k^x) \sigma_{x,m}(k) j^m J_{m,x}(k, \hat{\tau}_i) J_{m,x}(p, \hat{\tau}_i) g_0(\hat{\tau}_i) \end{aligned}$$

- ◆ Taking into account all the modes  $m$ , this leads to

$$\begin{aligned} & \left[ \omega_c - \omega_{x0} - m \omega_s \right] \hat{x}_m(\hat{\tau}_i) J_{m,x}(p, \hat{\tau}_i) g_0(\hat{\tau}_i) \\ &= -\frac{j e \pi I_b}{2 m_0 c \gamma Q_{x0}} \sum_k Z_x(\omega_k^x) \sigma_x(k) j^m J_{m,x}(k, \hat{\tau}_i) J_{m,x}(p, \hat{\tau}_i) g_0(\hat{\tau}_i) \end{aligned}$$

with

$$\sigma_x(k) = \sum_m \sigma_{x,m}(k)$$

- ◆ Multiplying by

$$-\frac{j^{-m} \hat{\tau}_i}{\pi}$$

and integrating, yields

$$\begin{aligned} & \left[ \omega_c - \omega_{x0} - m\omega_s \right] \sigma_{x,m}(p) \\ &= \frac{jeI_b}{2m_0 c \gamma Q_{x0}} \sum_k Z_x(\omega_k^x) \sigma_x(k) \int_0^{+\infty} J_{m,x}(k, \hat{\tau}_i) J_{m,x}(p, \hat{\tau}_i) g_0(\hat{\tau}_i) \hat{\tau}_i d\hat{\tau}_i \end{aligned}$$

- ◆ The previous low-intensity coherent modes of oscillation are recovered when only 1 value of  $m$  is considered
- ◆ For the general case (i.e. considering all the modes  $m$ ), one method consists in dividing the previous equation by  $\omega_c - \omega_{x0} - m\omega_s$  and integrating over  $m$

$$\sigma_x(p) = \frac{jeI_b}{2m_0 c \gamma Q_{x0}} \sum_k Z_x(\omega_k^x) \sigma_x(k) \sum_m \frac{1}{\omega_c - \omega_{x0} - m\omega_s} \int_0^{+\infty} J_{m,x}(k, \hat{\tau}_i) J_{m,x}(p, \hat{\tau}_i) g_0(\hat{\tau}_i) \hat{\tau}_i d\hat{\tau}_i$$

- ◆ Using the matrix notation, it can be written

$$\sigma_x(p) = \varepsilon \sum_k j Z_x(\omega_k^x) M_{pk} \sigma_x(k)$$

with, when assuming a water-bag bunch,

$$M_{pk} = 2 B \sum_m \frac{1}{\frac{\omega_c - \omega_{x0}}{\omega_s} - m} \int_0^1 J_{m,x}\left(k, \frac{\tau_b}{2} u\right) J_{m,x}\left(p, \frac{\tau_b}{2} u\right) u du$$

$$\varepsilon = \frac{e I_b}{4 \pi \gamma m_0 c Q_{x0} B \omega_s}$$

$$B = f_0 \tau_b$$

## TMCI (5/40)

### ◆ Method to solve this equation

- Assume a real coherent betatron frequency shift measured in incoherent synchrotron frequency unit

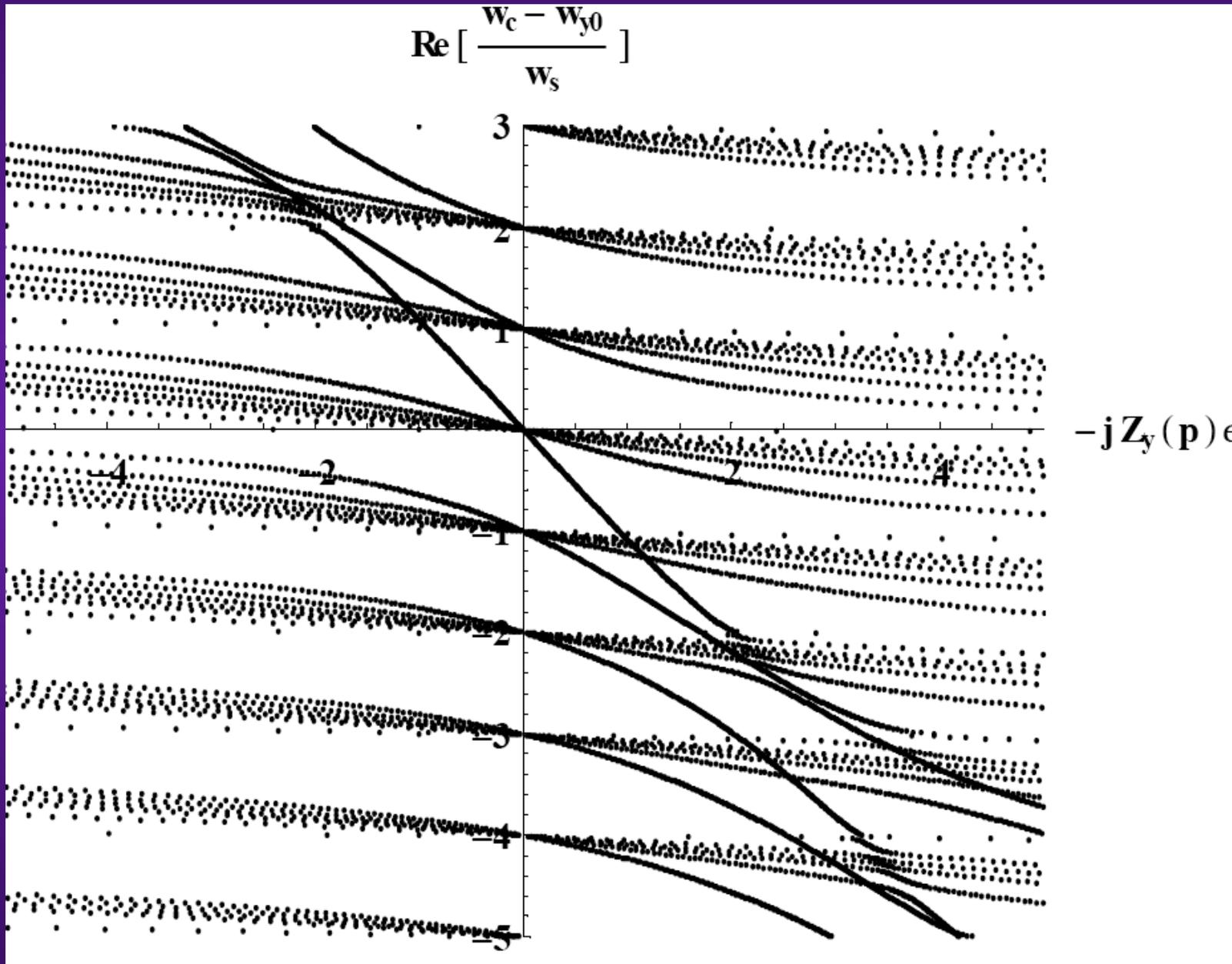
$$\frac{\omega_c - \omega_{x0}}{\omega_s}$$

- Look for the eigenvalues of the matrix

$$\left[ j Z_x(\omega_k^x) \right] \left[ M_{pk} \right]$$

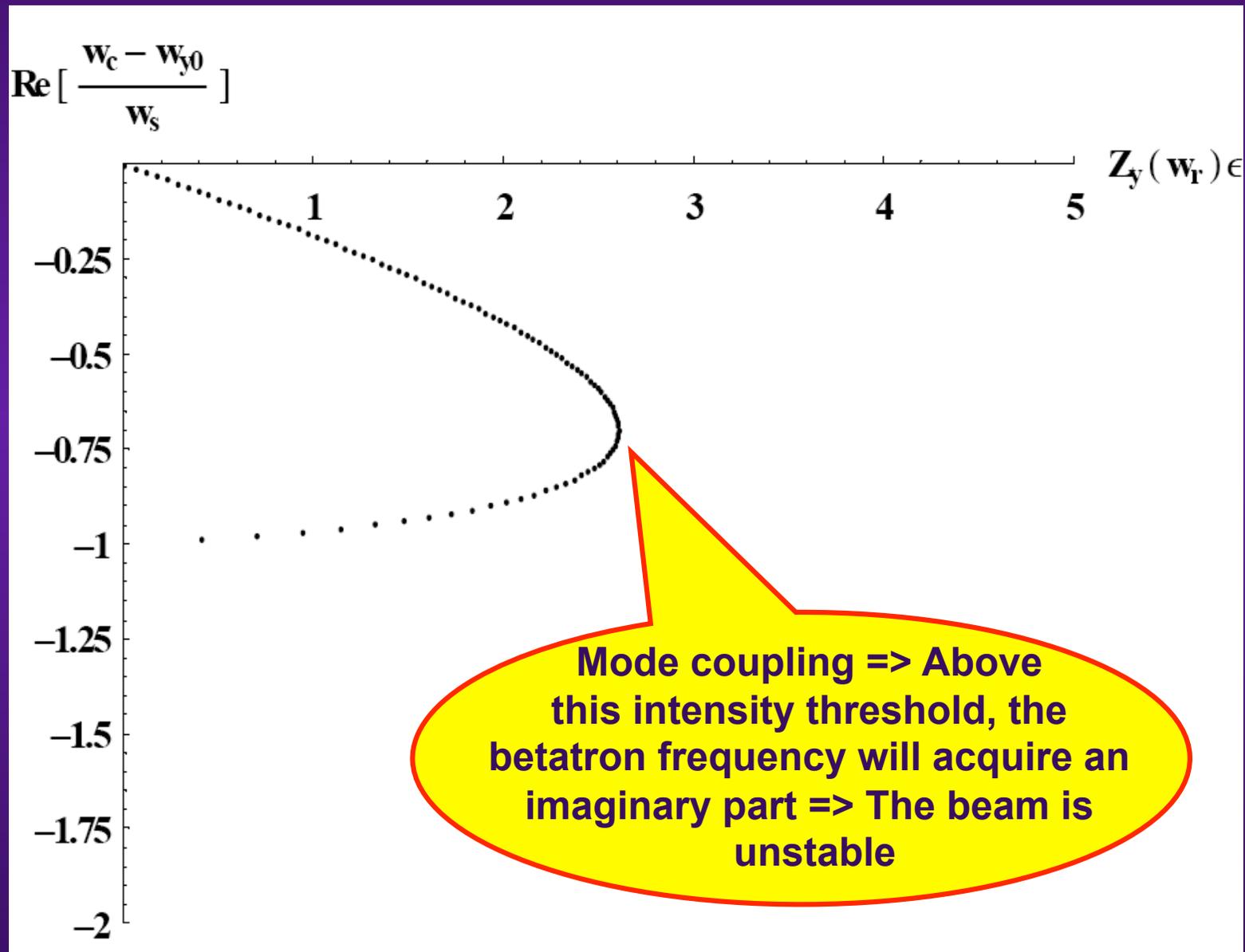
- Scale the intensity parameter  $\varepsilon$  in order to adjust the eigenvalue to unity
- Examples are given in the next slides

◆ 1) Constant inductive impedance



$$f_r \tau_b = 0.2 \quad \text{TMCI (7/40)}$$

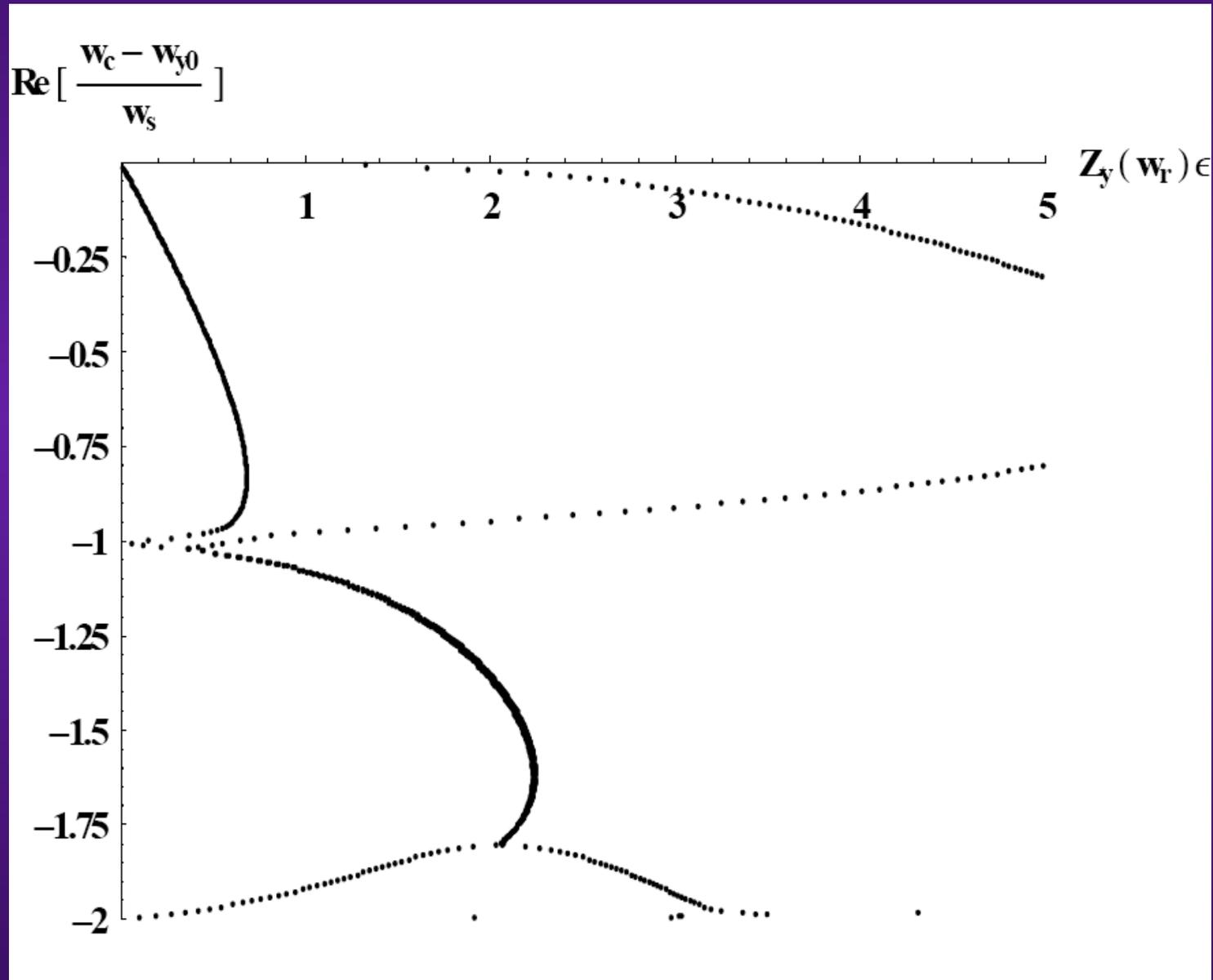
◆ 2) Very short bunch interacting with a Broad-Band impedance



$$f_r \tau_b = 0.8$$

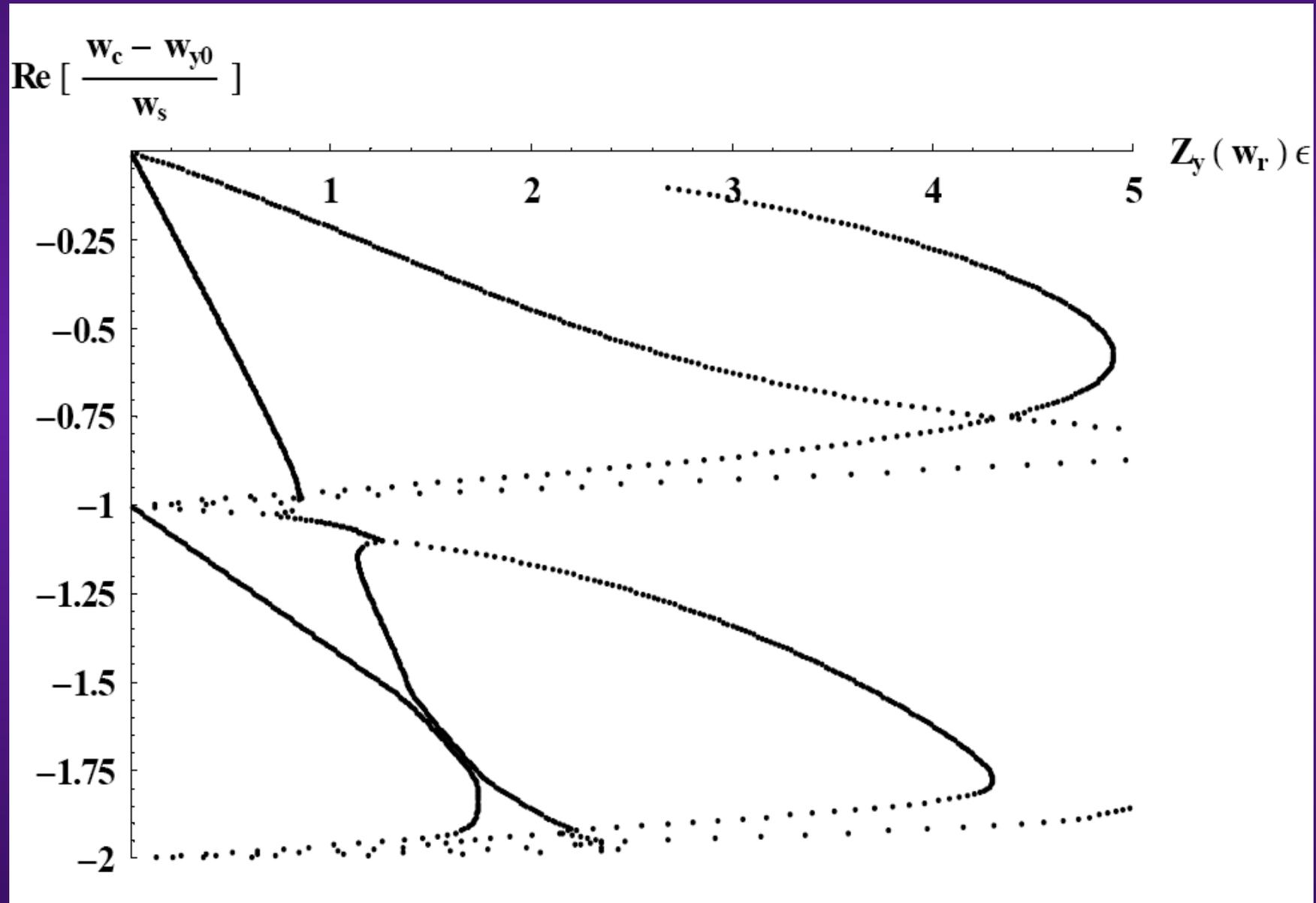
TMCI (8/40)

◆ 3) Short bunch interacting with a Broad-Band impedance



$f_r \tau_b = 1.9$  TMCI (9/40)

◆ 4) Long bunch interacting with a Broad-Band impedance



## TMCI (10/40)

- ◆ Using the (approximate) sinusoidal modes discussed before

$$p_m(t) = \begin{cases} \cos\left[ (|m|+1) \pi t / \tau_b \right], & m \text{ even} \\ \sin\left[ (|m|+1) \pi t / \tau_b \right], & m \text{ odd} \end{cases}$$

and generalizing the bunch spectrum to any mode  $(m,n)$

$$h_{m,n}(\omega) = p_m^*(\omega) p_n(\omega)$$

$$p_m(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} p_m(t) e^{-j\omega t} dt$$

yields

## TMCI (11/40)

$$h_{m,n}(\omega) = \frac{\tau_b^2}{\pi^4} (|m|+1) \times (|n|+1) \times F_m^n \\ \times \left\{ (\omega \tau_b / \pi)^2 - (|m|+1)^2 \right\}^{-1} \times \left\{ (\omega \tau_b / \pi)^2 - (|n|+1)^2 \right\}^{-1}$$

with

$$F_{m \text{ even}}^{n \text{ even}} = (-1)^{(|m|+|n|)/2} \times \cos^2[\omega \tau_b / 2]$$

$$F_{m \text{ even}}^{n \text{ odd}} = \frac{(-1)^{(|m|+|n|+3)/2}}{2j} \times \sin[\omega \tau_b]$$

$$F_{m \text{ odd}}^{n \text{ even}} = \frac{(-1)^{(|m|+|n|+1)/2}}{2j} \times \sin[\omega \tau_b]$$

$$F_{m \text{ odd}}^{n \text{ odd}} = (-1)^{(|m|+|n|+2)/2} \times \sin^2[\omega \tau_b / 2]$$

- ◆ The generalized Sacherer's formula for any mode  $(m,n)$  is then written

$$\Delta\omega_{m,n}^x = \left( |m| + 1 \right)^{-1} \frac{j e \beta I_b}{2 m_0 \gamma Q_{x0} \Omega_0 L_b} \left( Z_x^{eff} \right)_{m,n}$$

$$\left( Z_x^{eff} \right)_{m,n} = \frac{\sum_{k=-\infty}^{k=+\infty} Z_x \left( \omega_k^x \right) h_{m,n} \left( \omega_k^x - \omega_{\xi_x} \right)}{\sum_{k=-\infty}^{k=+\infty} h_{m,m} \left( \omega_k^x - \omega_{\xi_x} \right)}$$

## TMCI (13/40)

- ◆ Considering the case where 2 adjacent head-tail modes ( $m$  and  $m+1$ ) undergo a coupled motion, the stability of a high-intensity single-bunch beam can be discussed using the following determinant, e.g. in the vertical plane

$$\begin{vmatrix} \omega_c - \omega_{y,m} & -\Delta\omega_{m,m+1}^y \\ -\Delta\omega_{m+1,m}^y & \omega_c - \omega_{y,m+1} \end{vmatrix} = 0$$

with  $\omega_{y,m} = \omega_{y0} + m\omega_s + \Delta\omega_{m,m}^y$

- ◆ Remarks concerning  $h_{m,m+1}(\omega)$  : One can see that
  - It is a pure imaginary function
  - It is an odd function
  - $h_{m,m+1}(\omega) = -h_{m+1,m}(\omega)$

## TMCI (14/40)

$$\Rightarrow \Delta\omega_{m+1,m}^y = -k_m^2 \Delta\omega_{m,m+1}^y$$

$$k_m = \sqrt{\frac{|m|+1}{|m+1|+1}}$$

- ◆ Considering the case of a driving broad-band resonator, the coupling impedance is given by

$$Z_y(\omega) = \frac{\omega_r R_r}{\omega} \left[ 1 - j Q_r \left( \frac{\omega_r}{\omega} - \frac{\omega}{\omega_r} \right) \right]$$

- ◆ The following solutions are obtained

$$\omega_c^\pm = \frac{1}{2} \times \left[ 2\omega_{y0} + (2m+1)\omega_s + \Delta\omega_{m,m}^y + \Delta\omega_{m+1,m+1}^y \right] \\ \pm \frac{1}{2} \sqrt{\left( \omega_s + \Delta\omega_{m+1,m+1}^y - \Delta\omega_{m,m}^y \right)^2 - 4k_m^2 \left( \Delta\omega_{m,m+1}^y \right)^2}$$

◆ Writing

$$\Delta\omega_{m,m}^y = a_0 I_b$$

$$\Delta\omega_{m+1,m+1}^y = b_0 I_b$$

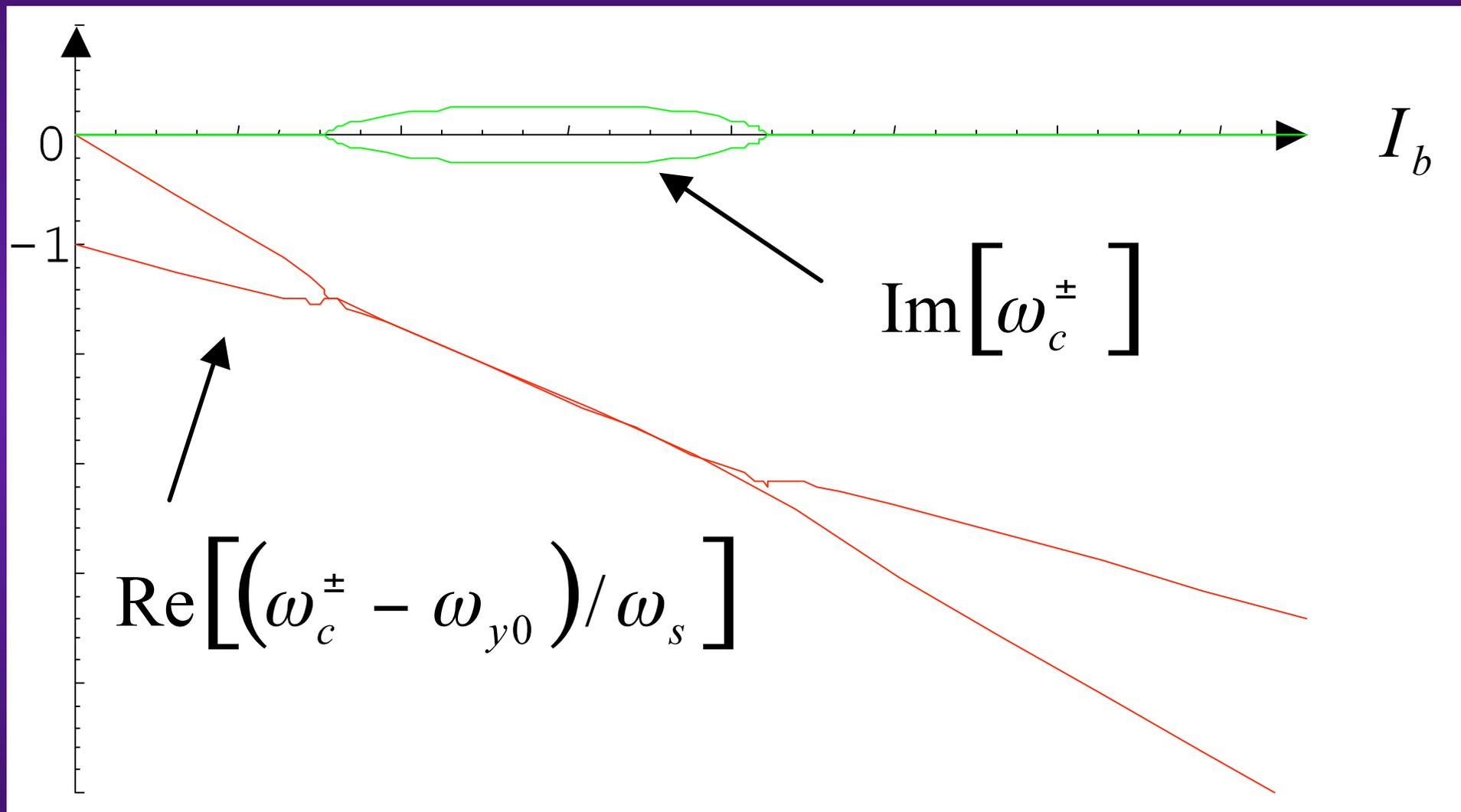
$$\Delta\omega_{m,m+1}^y = c_0 I_b$$

⇒

$$I_{b,th1} = \frac{\omega_s}{a_0 - b_0 + 2k_m |c_0|}$$

$$I_{b,th2} = \frac{\omega_s}{a_0 - b_0 - 2k_m |c_0|}$$

- ◆ If  $I_{b,th2} > 0$ , then  $I_{b,th2} > I_{b,th1}$ . The beam is stable from zero intensity to  $I_{b,th1}$ . Then it is unstable between  $I_{b,th1}$  and  $I_{b,th2}$  (mode-coupling at  $I_{b,th1}$ ). Finally, it is stable again above  $I_{b,th2}$  (mode-decoupling at  $I_{b,th2}$ ). This case is depicted in the next figure

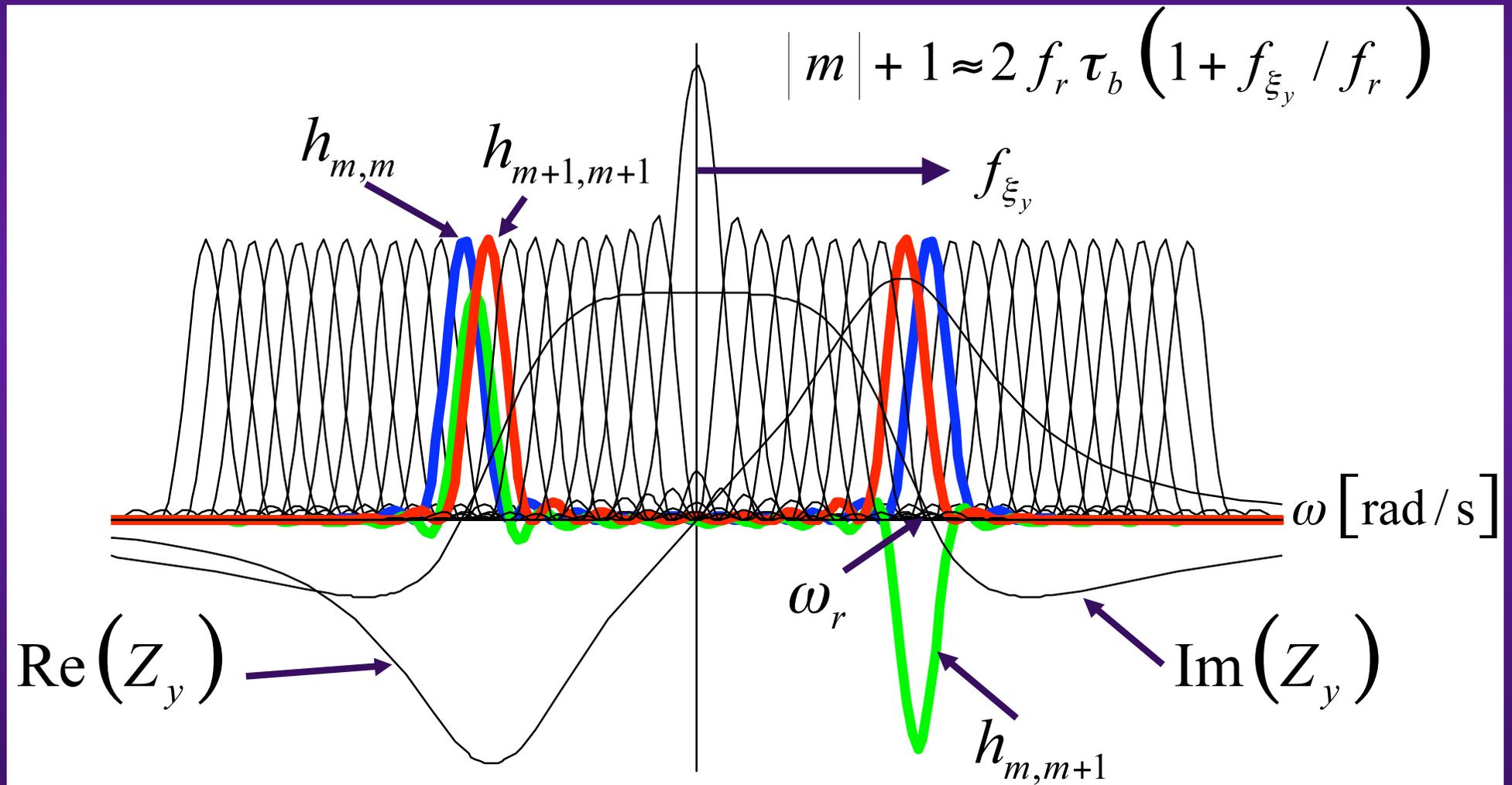


## TMCI (17/40)

- ◆ This corresponds to the case of a “long bunch” (with respect to the impedance:  $\tau_b \gg 0.5/f_r \Rightarrow$  See next slide), whose spectra of modes 0 and  $-1$  peak at low frequencies. Both modes couple to the inductive part of the coupling impedance, and therefore are shifted in the same direction. Moreover, their coupling to the resistive part of the coupling impedance is weak. As a consequence, when the two modes merge, they cannot develop a strong instability and are pulled apart as intensity increases. Modes of higher order can couple, but higher-order modes are more difficult to drive than lower-order ones and therefore the intensity threshold is expected to be higher

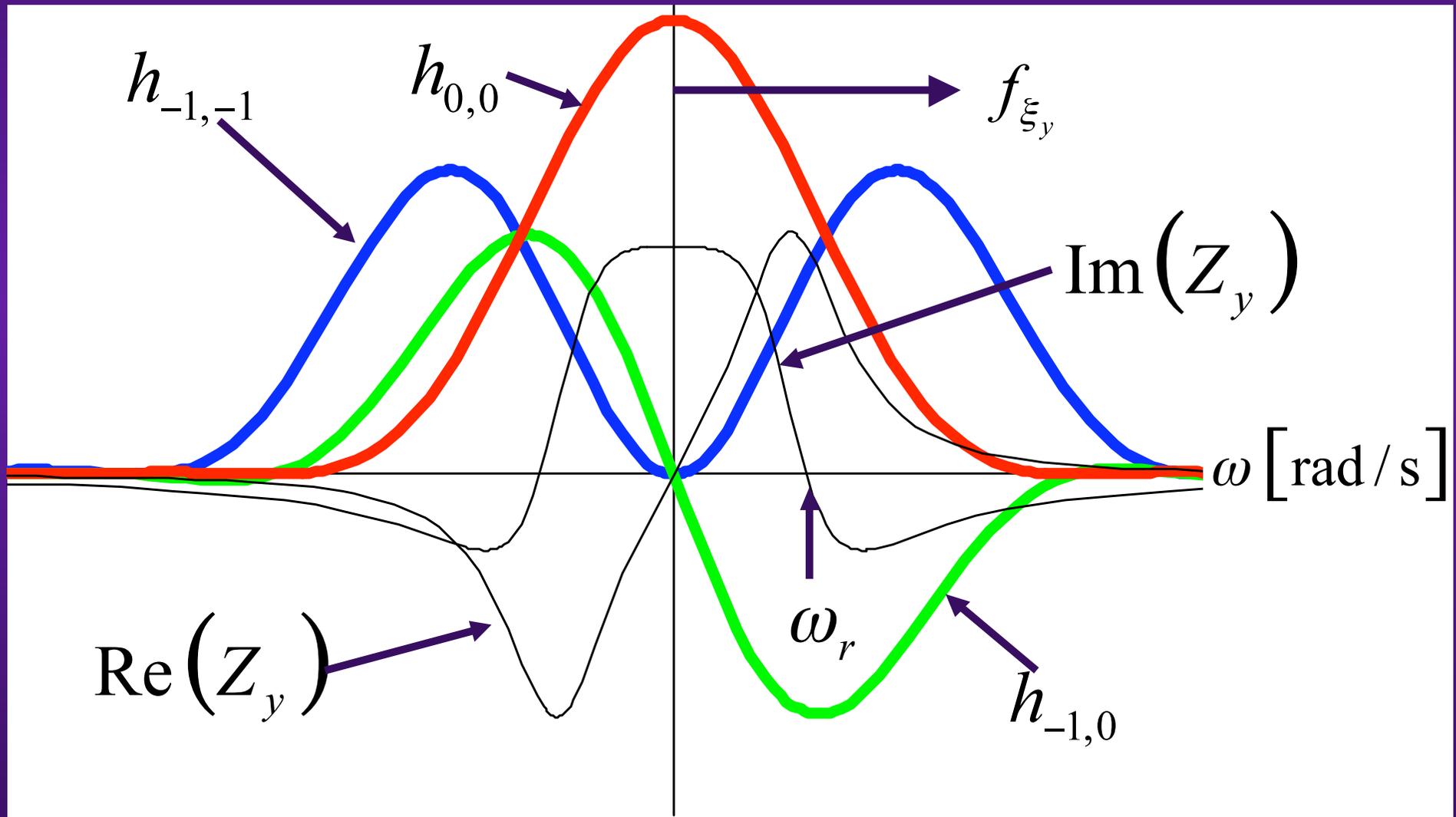
“Long-bunch” regime:

$$\tau_b \gg 0.5 / f_r$$



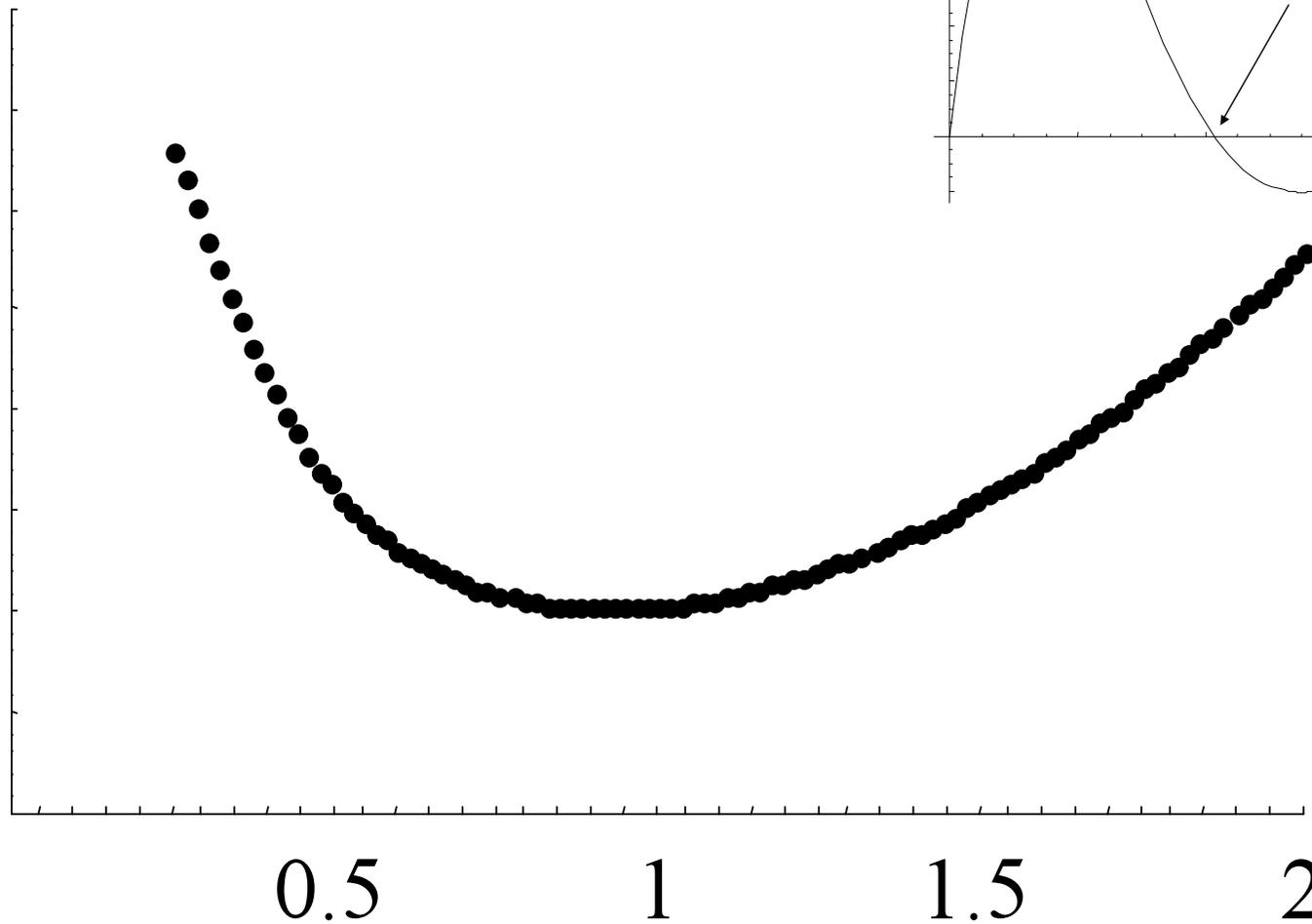
# TMCI (19/40)

$\tau_b = 0.5 / f_r$

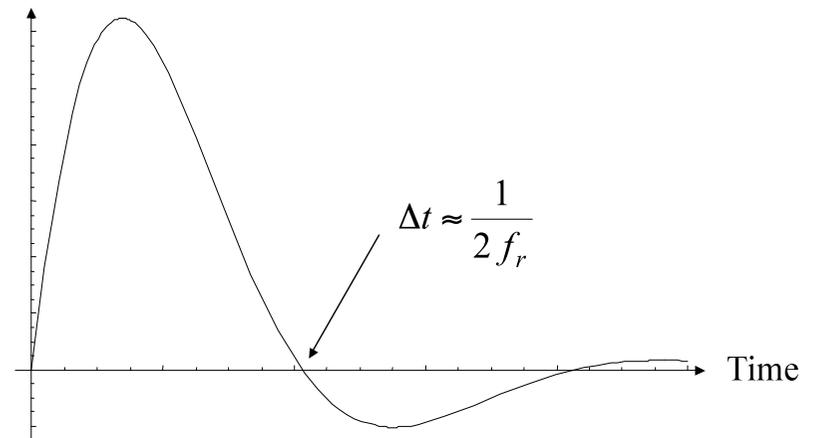


# TMCI (20/40)

$N_{b,th}$

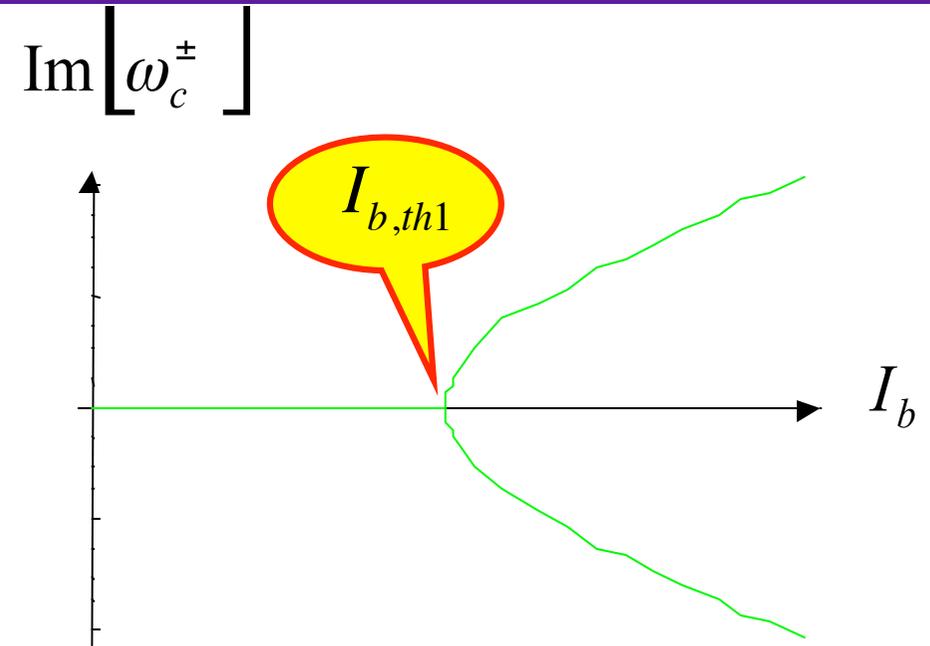
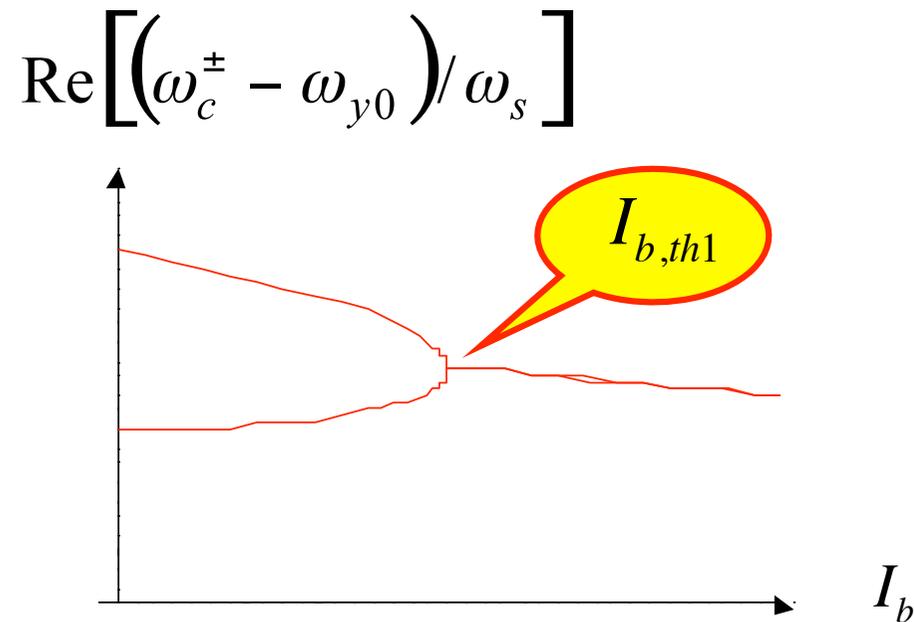


Transverse wake-field



## TMCI (21/40)

- ◆ In the following, one will consider for our model the mode-coupling between the two most critical head-tail modes ( $m$  and  $m+1$ ) overlapping the peak of the negative resistive impedance. In this case there will never be mode-decoupling (  $I_{b,th2} < 0$  ), and the threshold for mode-coupling is obtained at the intensity  $I_{b,th1}$



## TMCI (22/40)

- ◆ Below the intensity threshold, the real and imaginary parts of the coherent frequencies are given by

$$\begin{aligned} \operatorname{Re} \left( \omega_c^\pm \right) &= \omega_{y0} + (m + 1/2) \omega_s + I_b (a_0 + b_0) / 2 \\ &\pm \frac{1}{2} \sqrt{\left[ \omega_s + (b_0 - a_0) I_b \right]^2 - 4 k_m^2 c_0^2 I_b^2} \end{aligned}$$

$$\operatorname{Im} \left( \omega_c^\pm \right) = 0$$

- ◆ Above the intensity threshold, the real and imaginary parts of the coherent frequencies are given by

$$\operatorname{Re} \left( \omega_c^\pm \right) = \omega_{y0} + (m + 1/2) \omega_s + I_b (a_0 + b_0) / 2$$

$$\operatorname{Im} \left( \omega_c^\pm \right) = \pm \frac{1}{2} \sqrt{4 k_m^2 c_0^2 I_b^2 - \left[ \omega_s + (b_0 - a_0) I_b \right]^2}$$

- ◆ The instability rise-times are given by

$$\tau_{\pm} = \frac{-1}{\text{Im}(\omega_c^{\pm})}$$

which gives, for the unstable mode

$$\tau_{-} = T_s \times \frac{1}{\pi \sqrt{(\alpha - 1)(\alpha q + 1)}}$$

where

$$q = \frac{2k_m |c_0| + b_0 - a_0}{2k_m |c_0| - b_0 + a_0}$$

$$\alpha = \frac{I_b}{I_{b,th1}}$$

~ 1 for long bunches

- ◆ The intensity threshold can be found from the previous equation

$$\text{Im} \left( \omega_c^\pm \right) = \pm \frac{1}{2} \sqrt{4 k_m^2 c_0^2 I_b^2 - \left[ \omega_s + (b_0 - a_0) I_b \right]^2}$$

- ◆ It is given by  $4 k_m^2 c_0^2 I_{b,th}^2 = \left[ \omega_s + (b_0 - a_0) I_b \right]^2$

- ◆ In the case of a long bunch (see figure)  $b_0 \approx a_0 \approx 0$

$$\Rightarrow I_{b,th} \approx \frac{\omega_s}{2 k_m |c_0|}$$

◆ This leads to

$$N_{b,th} = \frac{4 \pi^3 f_s Q_{y0} E \tau_b^2}{e c} \times \frac{f_r}{|Z_y|} \times \left( 1 + \frac{f_{\xi_y}}{f_r} \right)$$

which can be re-written

$$N_{b,th} = \frac{8 \pi Q_{y0} |\eta| \varepsilon_l}{e \beta^2 c} \times \frac{f_r}{|Z_y|} \times \left( 1 + \frac{f_{\xi_y}}{f_r} \right)$$

using

$$f_s = |\eta| \times \left( \Delta p / p_0 \right)_{\max} / \left( \pi \tau_b \right)$$

$$\varepsilon_l = \beta^2 E \tau_b \left( \Delta p / p_0 \right)_{\max} \pi / 2$$

~ Same result as  
for coasting beams!  
(within a factor ~ 2)

## TMCI (26/40)

- ◆ Furthermore, when bunch

$$\alpha = \frac{I_b}{I_{b,th1}} \gg 1$$

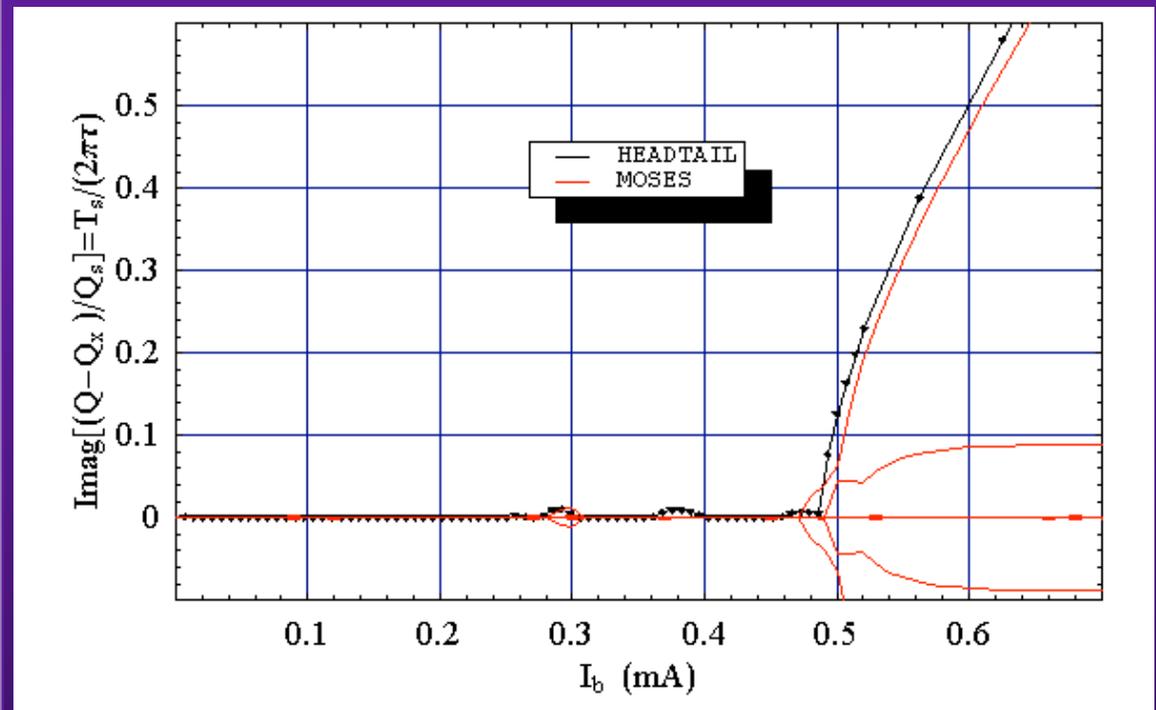
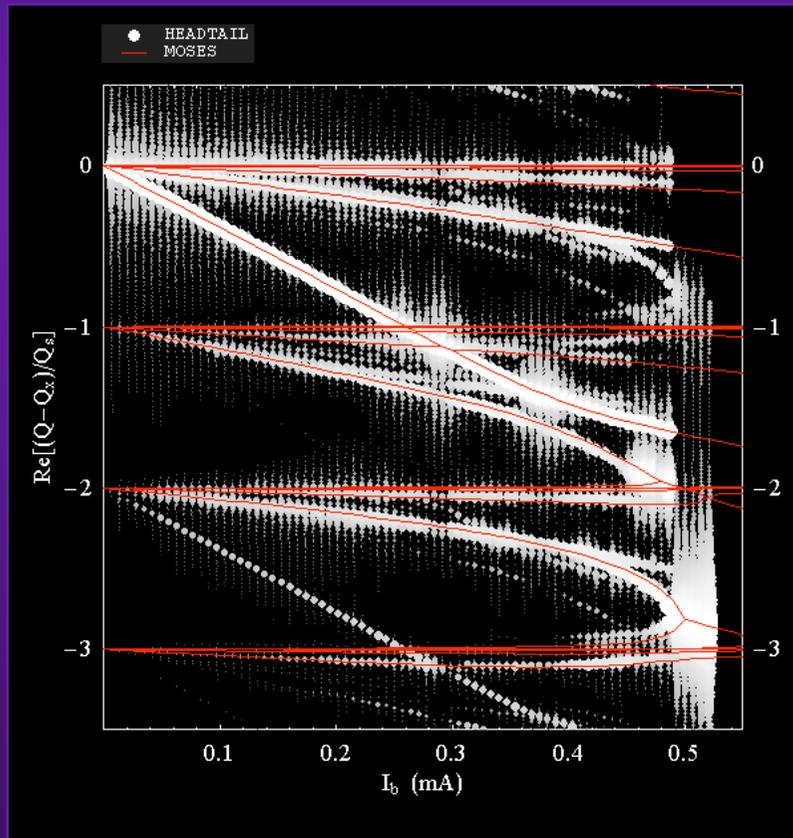
, and in the case of a long

$$\tau_- \approx \frac{T_s N_{b,th}}{\pi N_b}$$

As  $N_{b,th} \propto \frac{1}{T_s}$   $\Rightarrow$  The instability rise-time becomes independent of synchrotron motion as could be anticipated (as the instability rise-time is much faster than synchrotron period)

## TMCI (27/40)

- ◆ This can be checked with the MOSES code, which is a program computing the coherent bunched-beam mode
- ◆ Below is a comparison between MOSES code and the HEADTAIL code, which is a code simulating single-bunch phenomena, in the case of a LHC-type single bunch at SPS injection



*Courtesy of B. Salvant*

# BB resonator impedance

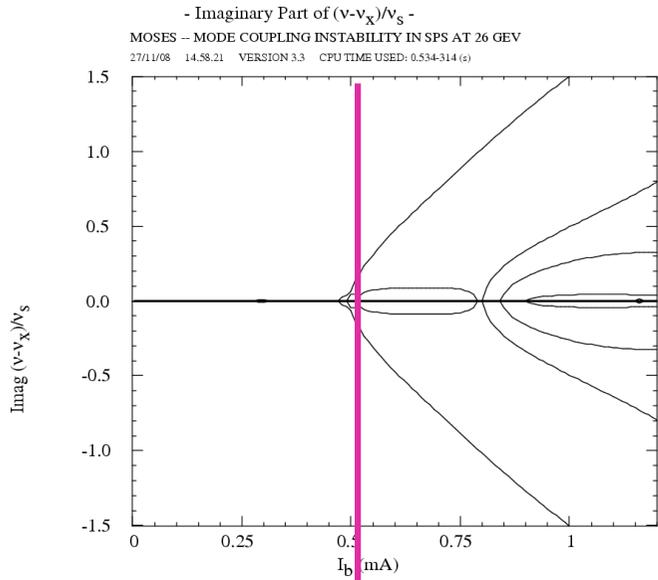
# MOSES

$$R_y = 10 \text{ M}\Omega/\text{m}$$

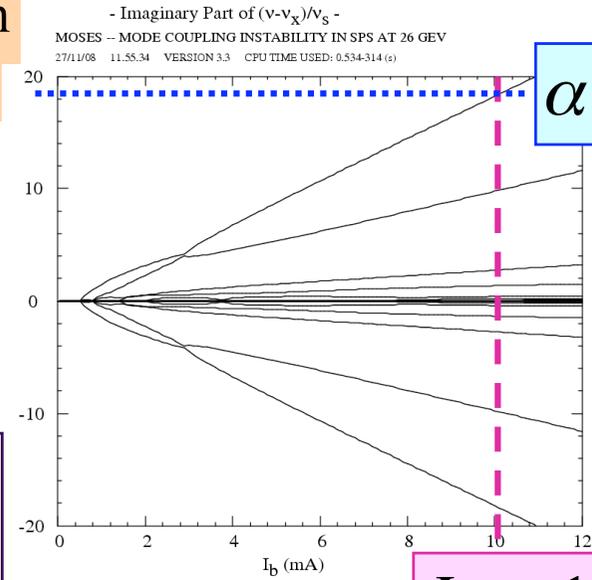
$$f_r = 1 \text{ GHz}$$

$$Q = 1$$

$$\tau_{\text{TMC}}^{\text{MOSES}} = \frac{T_s}{2\pi\alpha}$$



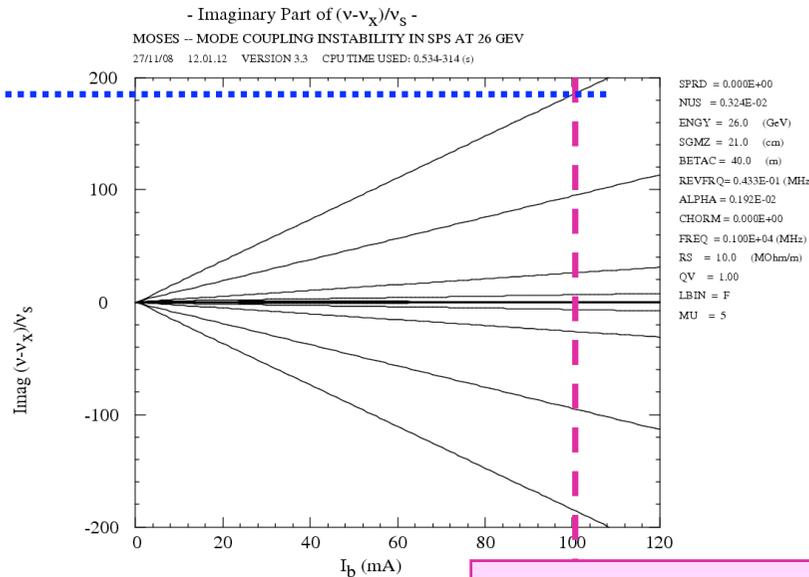
$$I_b^{\text{th}} \approx 0.5 \text{ mA}$$



$$\alpha_1 \approx 18.5$$

$$I_{b1} = 10 \text{ mA}$$

$$\alpha_2 \approx 185$$



$$\frac{I_{b2}}{I_{b1}} = 10$$

$$\frac{\alpha_2}{\alpha_1} \approx \frac{185}{18.5} = 10$$

$$I_{b2} = 100 \text{ mA}$$

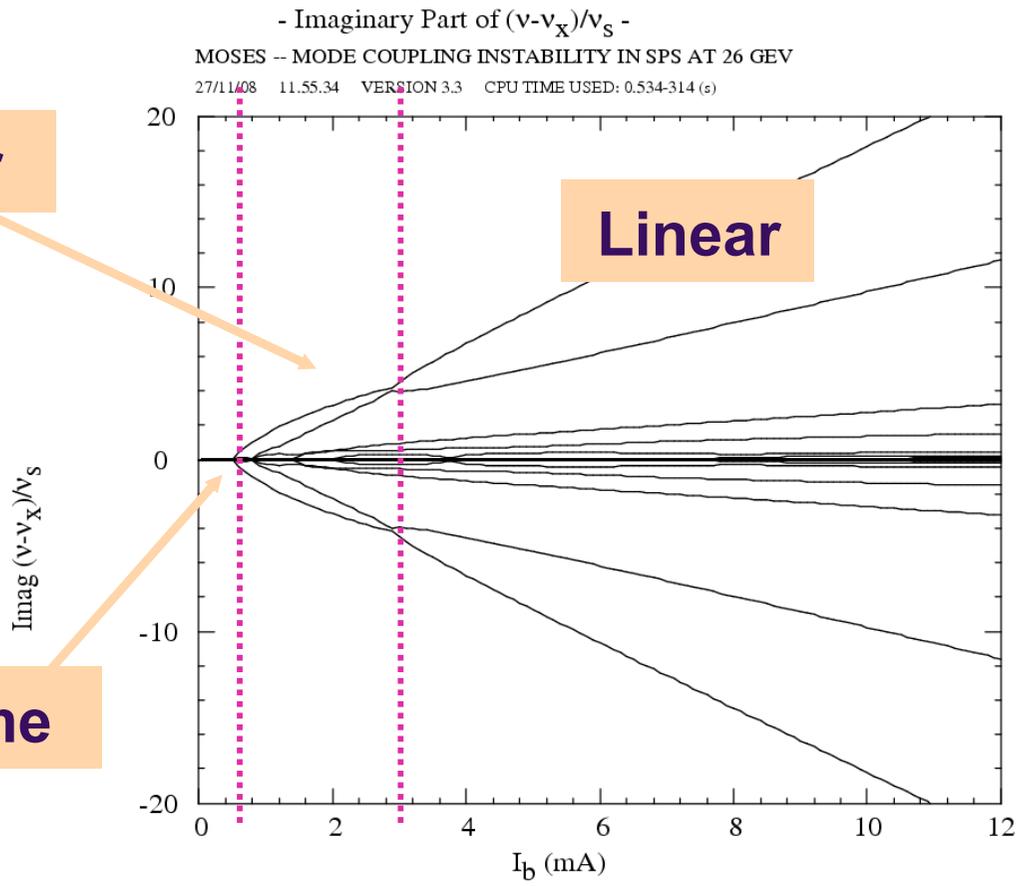
$$\Rightarrow \tau_{\text{TMC}}^{\text{MOSES}} \propto \frac{T_s}{2\pi I_b}$$

## ◆ General picture (for 0 chromaticity)

**Nonlinear**

**Linear**

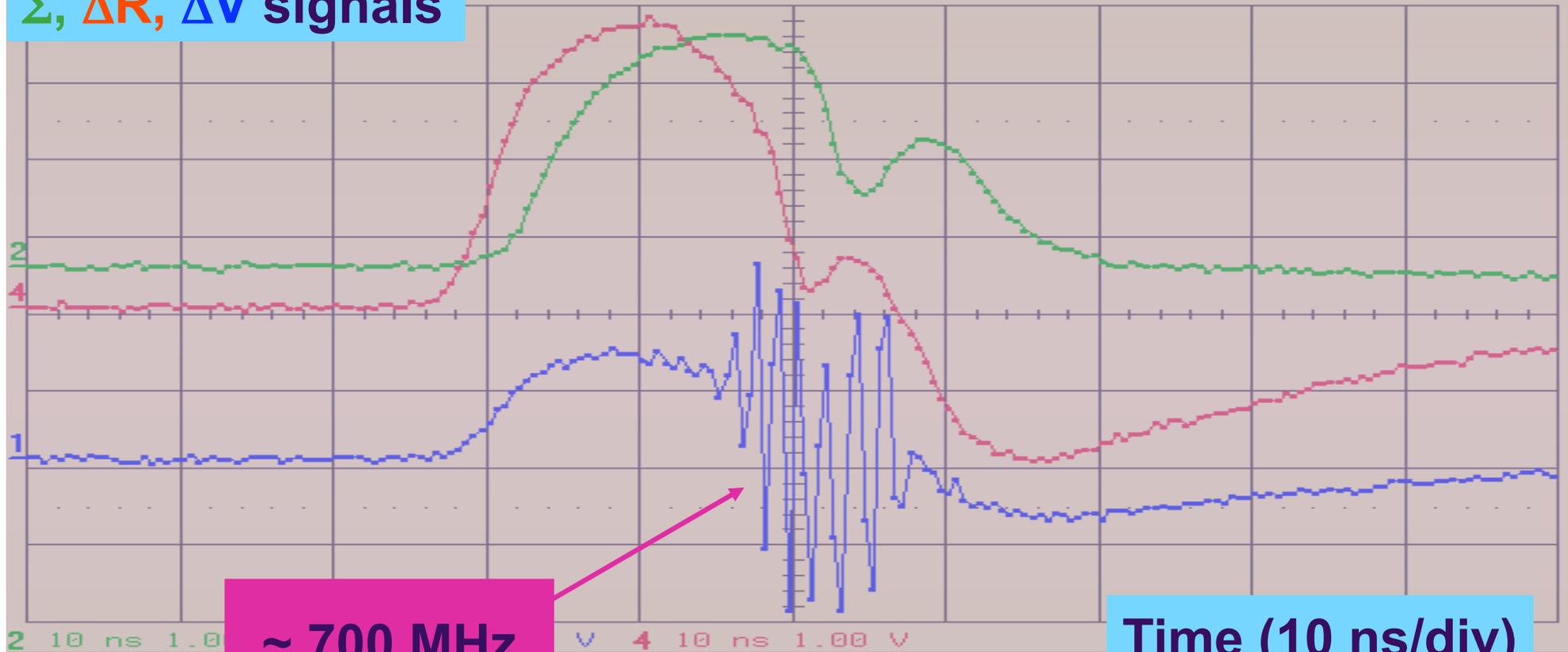
**Infinite rise-time**



# TMCI (30/40)

## ◆ PS measurements near transition

$\Sigma$ ,  $\Delta R$ ,  $\Delta V$  signals



**Instability suppressed by increasing the longitudinal emittance**

# TMCI (31/40)

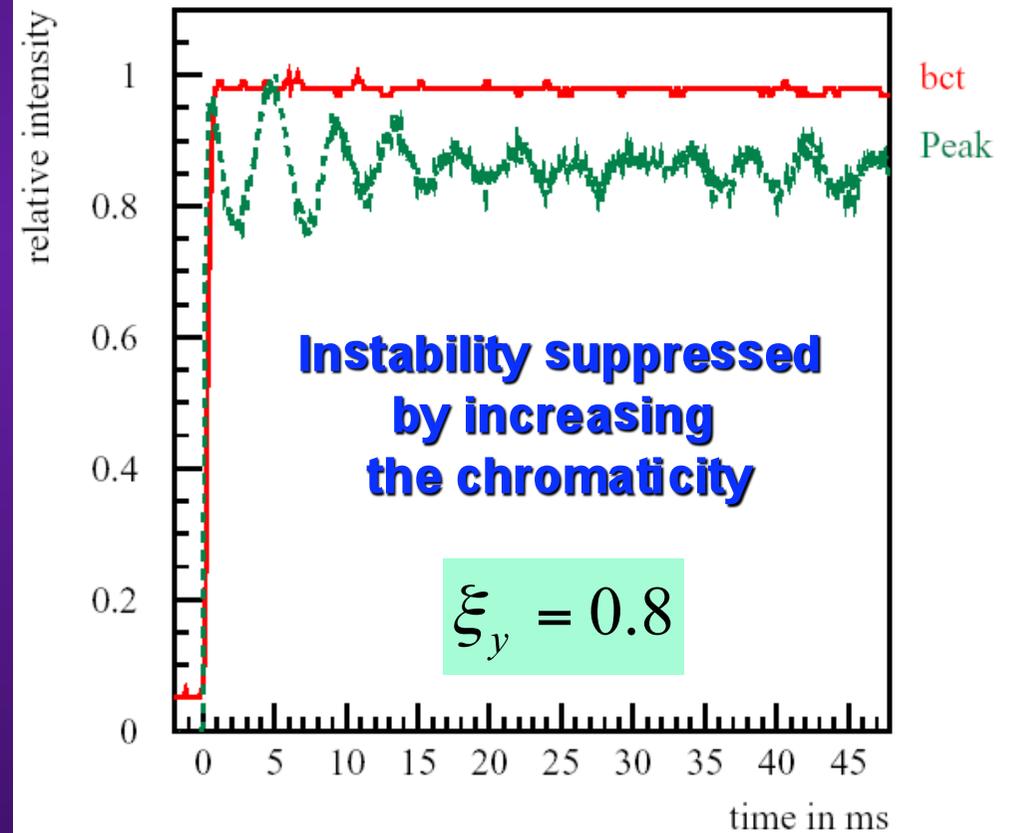
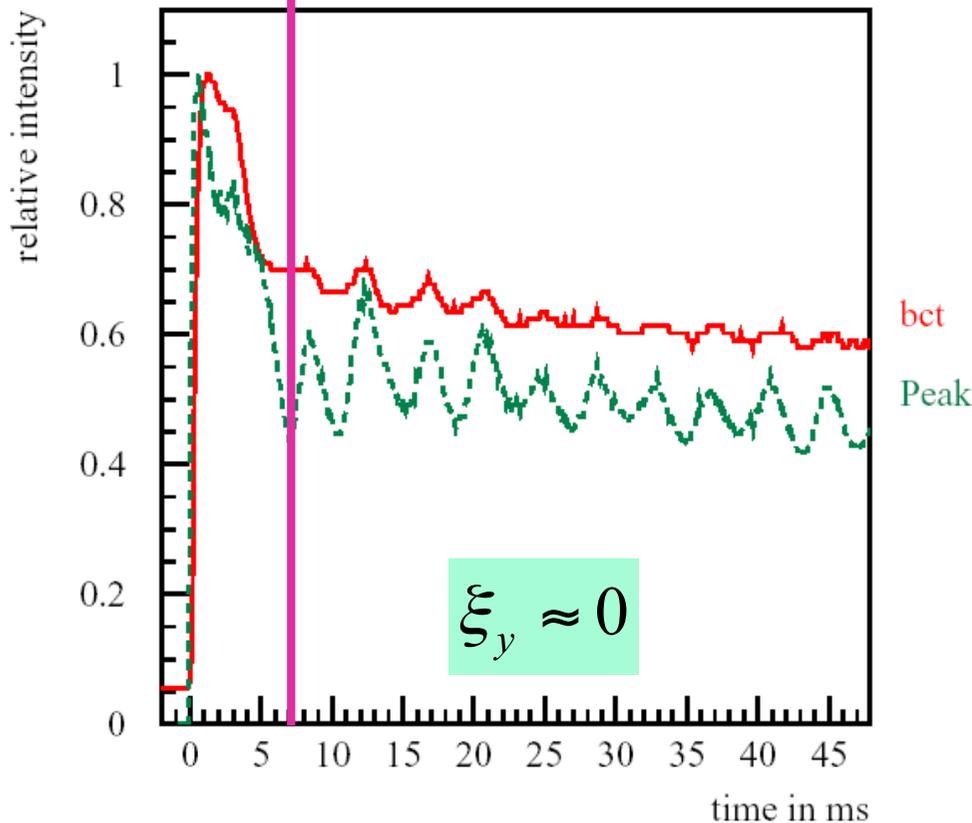
- ◆ SPS measurements at injection => See also the Movie for the case of a CERN SPS LHC-type bunch (under Windows!)

$$p = 26 \text{ GeV}/c$$

$$N_b \approx 1.2 \cdot 10^{11} \text{ p/b}$$

Synchrotron period  $\approx 7 \text{ ms}$

$$\varepsilon_l \approx 0.2 \text{ eVs} < \varepsilon_l^{\text{LHC}} = 0.35 \text{ eVs}$$

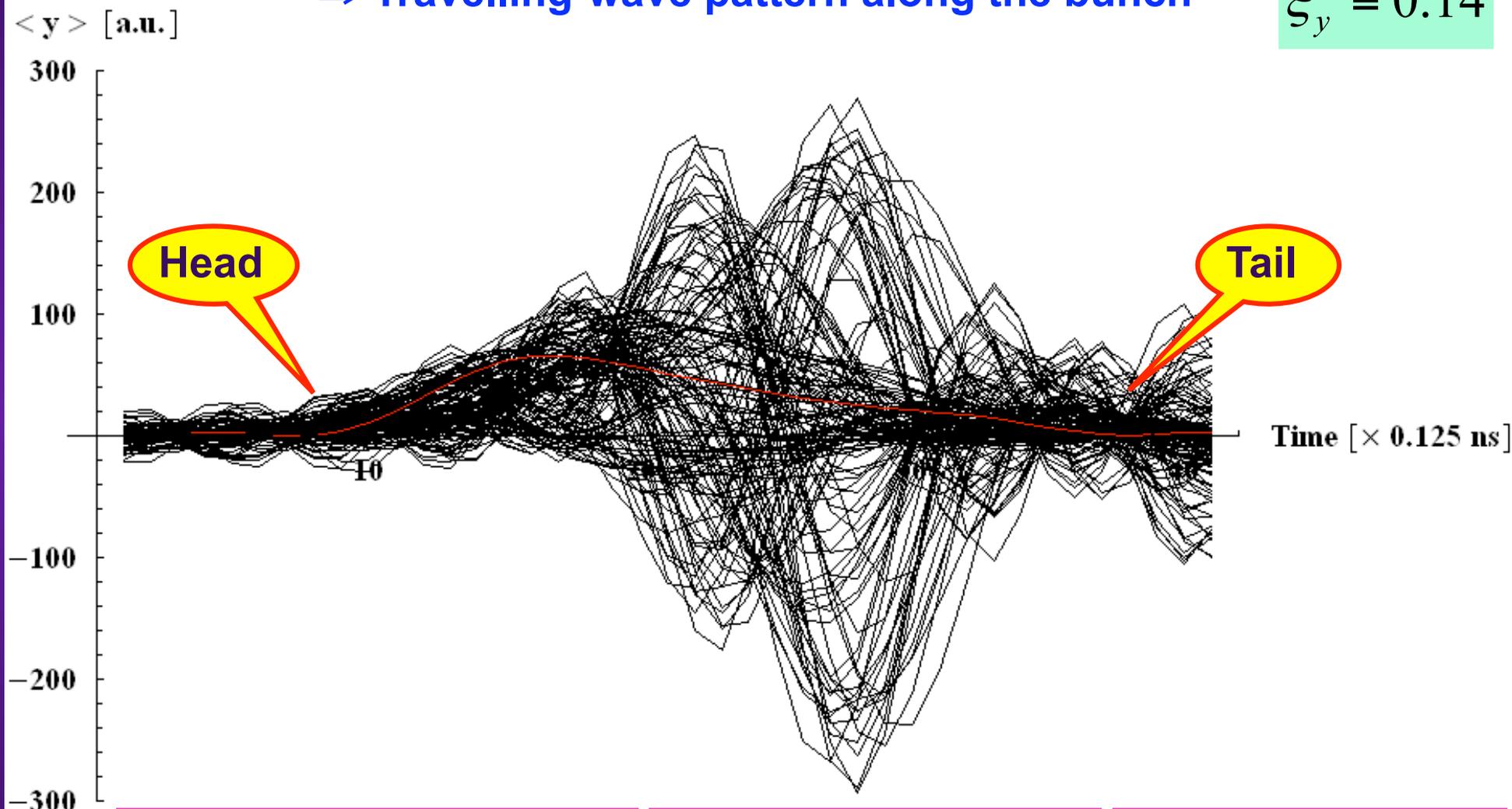


$$T_{rev}^{SPS} \approx 23 \mu\text{s}$$

# TMCI (32/40)

⇒ Travelling-wave pattern along the bunch

$$\xi_y = 0.14$$



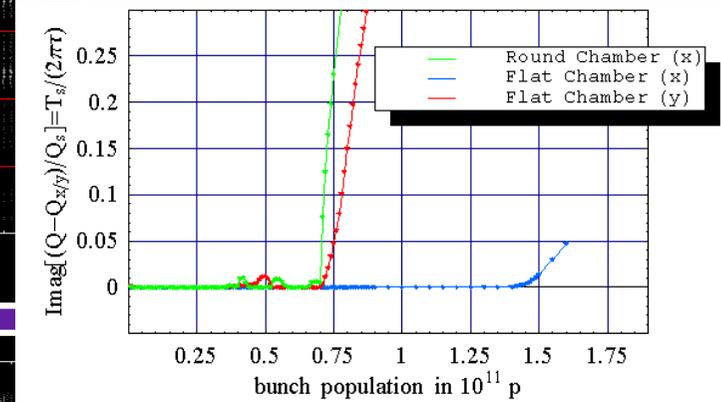
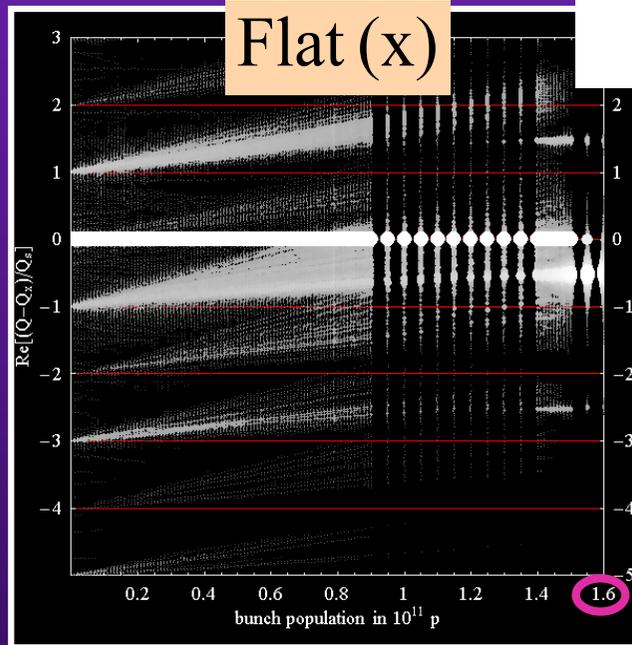
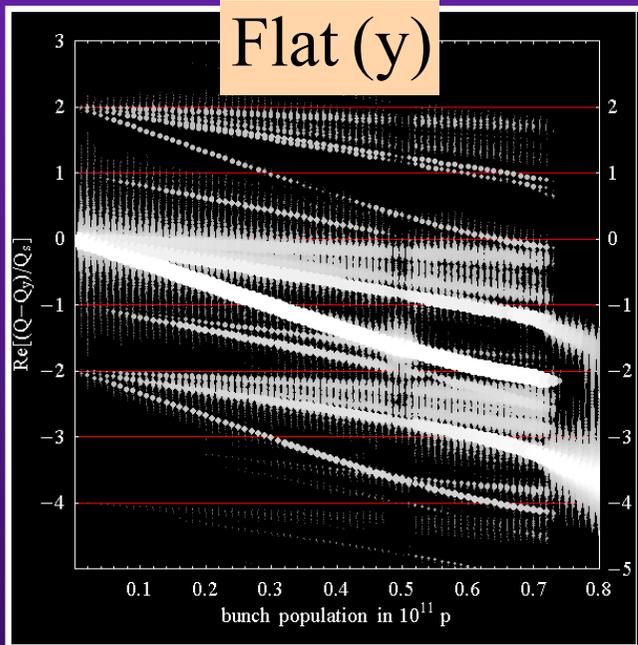
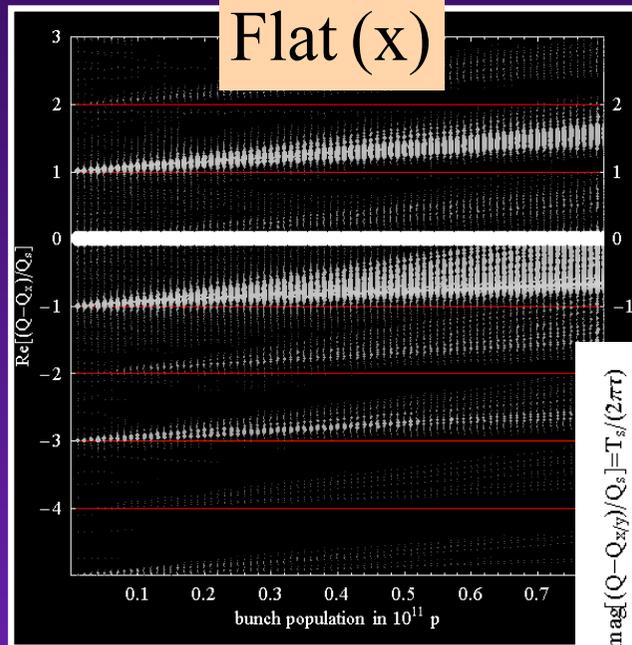
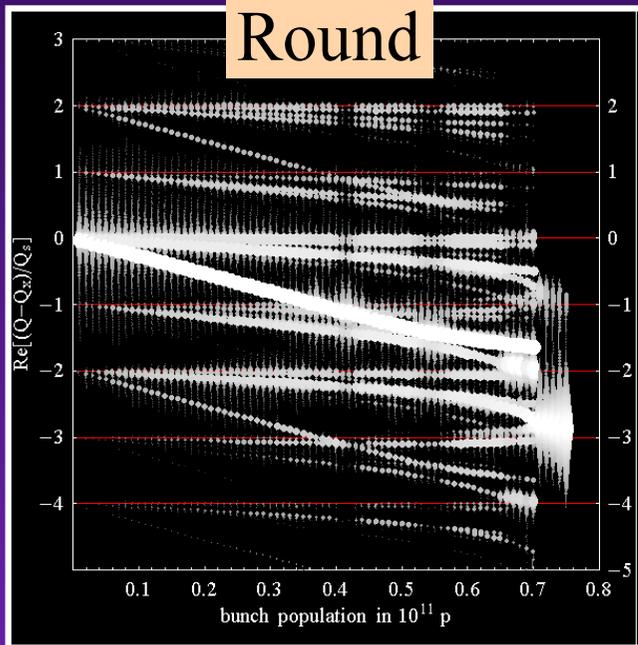
1<sup>st</sup> trace (in red) = turn 2

Last trace = turn 150

Every turn shown

# TMCI (33/40)

◆ Effect of flat chamber (in the case of the SPS)



Courtesy of B. Salvant

## TMCI (34/40)

- ◆ Effect of linear coupling between the transverse planes in the case of an asymmetric (flat) chamber
- ◆ => Using the same formalism as before, i.e. considering the case where 2 adjacent head-tail modes ( $m$  and  $m+1$ ) undergo a coupled motion, the new system to solve is

(near  $Q_x - Q_y = l$  )

Same notation as when it was discussed with coasting beams

$\omega_c - \omega_{x,m}$	$-\Delta\omega_{m,m+1}^x$	$-\frac{\hat{K}_0(l) R^2 \Omega_0^2}{2 \omega_{x0}}$	$0$	=0
$-\Delta\omega_{m+1,m}^x$	$\omega_c - \omega_{x,m+1}$	$0$	$-\frac{\hat{K}_0(l) R^2 \Omega_0^2}{2 \omega_{x0}}$	
$-\frac{\hat{K}_0(-l) R^2 \Omega_0^2}{2 \omega_{y0}}$	$0$	$\omega_c - \omega_{y,m}$	$-\Delta\omega_{m,m+1}^y$	
$0$	$-\frac{\hat{K}_0(-l) R^2 \Omega_0^2}{2 \omega_{y0}}$	$-\Delta\omega_{m+1,m}^y$	$\omega_c - \omega_{y,m+1}$	

## TMCI (35/40)

with  $\omega_{x,m} = \omega_{x0} + m\omega_s + \Delta\omega_{m,m}^x$        $\omega_{y,m} = \omega_{y0} + l\Omega_0 + m\omega_s + \Delta\omega_{m,m}^y$

This leads to a 4<sup>th</sup> order equation, which can be written

$$\left[ \left( \omega_c - \omega_{x,m} \right) \left( \omega_c - \omega_{x,m+1} \right) + \left( \Delta\omega_{m,m+1}^x \right)^2 \right] \times$$

$$\left[ \left( \omega_c - \omega_{y,m} \right) \left( \omega_c - \omega_{y,m+1} \right) + \left( \Delta\omega_{m,m+1}^y \right)^2 \right] = \frac{\left| \hat{\underline{K}}_0(l) \right|^2 R^4 \Omega_0^4}{4 \omega_{x0} \omega_{y0}} \times$$

$$\left[ \left( \omega_c - \omega_{x,m} \right) \left( \omega_c - \omega_{y,m} \right) + \left( \omega_c - \omega_{x,m+1} \right) \left( \omega_c - \omega_{y,m+1} \right) \right]$$

$$\left[ - \frac{\left| \hat{\underline{K}}_0(l) \right|^2 R^4 \Omega_0^4}{4 \omega_{x0} \omega_{y0}} - 2 \Delta\omega_{m,m+1}^x \Delta\omega_{m,m+1}^y \right]$$

This equation can be solved on the resonance (using here the approximation  $k_m = 1$  )

$$Q_{x0} + \frac{1}{2\Omega_0} \left( \Delta\omega_{m,m}^x + \Delta\omega_{m+1,m+1}^x \right) = Q_{y0} + l + \frac{1}{2\Omega_0} \left( \Delta\omega_{m,m}^y + \Delta\omega_{m+1,m+1}^y \right)$$

A necessary condition for stability is given by

$$\left| \Delta\omega_{m,m+1}^x + \Delta\omega_{m,m+1}^y \right| \leq \frac{1}{2} \left| 2\omega_s + \Delta\omega_{m+1,m+1}^x + \Delta\omega_{m+1,m+1}^y - \Delta\omega_{m,m}^x - \Delta\omega_{m,m}^y \right|$$

If the previous equation is fulfilled, then it is possible to stabilise the beam by linear coupling. Beam stability is obtained above a certain threshold for the coupling strength, whose value is given by

$$\begin{aligned}
 \left| \hat{K}_0(l) \right| &\geq \frac{2\sqrt{Q_{x0} Q_{y0}}}{R^2 \Omega_0} \\
 &\times \left[ \frac{1}{2} \left( \Delta\omega_{m,m}^x - \omega_s - \Delta\omega_{m+1,m+1}^x \right) \mp \Delta\omega_{m,m+1}^x \right]^{1/2} \\
 &\times \left[ \frac{1}{2} \left( \Delta\omega_{m,m}^y - \omega_s - \Delta\omega_{m+1,m+1}^y \right) \pm \Delta\omega_{m,m+1}^y \right]^{1/2}
 \end{aligned}$$

Consider for instance the case where  $\xi_x = \xi_y$ ,  $Q_x = Q_y$  and  $Z_y = \lambda Z_x$   
 The necessary condition for stability becomes

$$\left| \Delta\omega_{m,m+1}^y \right| \leq \frac{1}{2} \left| \omega'_s + \Delta\omega_{m+1,m+1}^y - \Delta\omega_{m,m}^y \right|$$

## TMCI (38/40)

This is the one-dimensional vertical stability criterion with the angular synchrotron frequency  $\omega_s$  replaced by

$$\omega'_s = \omega_s \times \frac{2\lambda}{\lambda+1}$$

If  $\lambda \gg 1$   $\Rightarrow$  A factor 2 can be gained on the TMCI intensity threshold

If  $\lambda = 2$   $\Rightarrow$  A factor 4/3 (i.e. 33%) can be gained on the TMCI intensity threshold

The last case was checked with the HEADTAIL code and a good agreement was found

◆ **Round chamber**

$$R_y = 20 \text{ M}\Omega/\text{m}$$

$$N_b^{th,x} = N_b^{th,y} \approx 2.8 \times 10^{10} \text{ p / b}$$

y-Yokoya factor

◆ **Flat chamber**

$$N_b^{th,y} \approx 3.3 \times 10^{10} \text{ p / b}$$

$$2.8 \times \frac{12}{\pi^2} = 3.4$$

$$N_b^{th,x} \approx 8 \times 10^{10} \text{ p / b}$$

$$2.8 \times \frac{24}{\pi^2} = 6.8$$

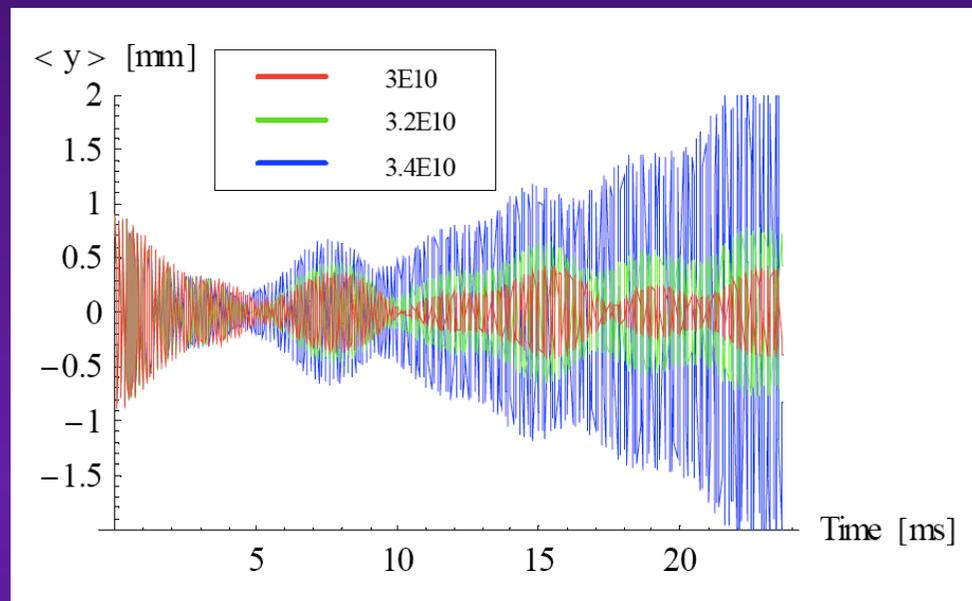
x-Yokoya factor

⇒ The intensity threshold is increased in a flat chamber by

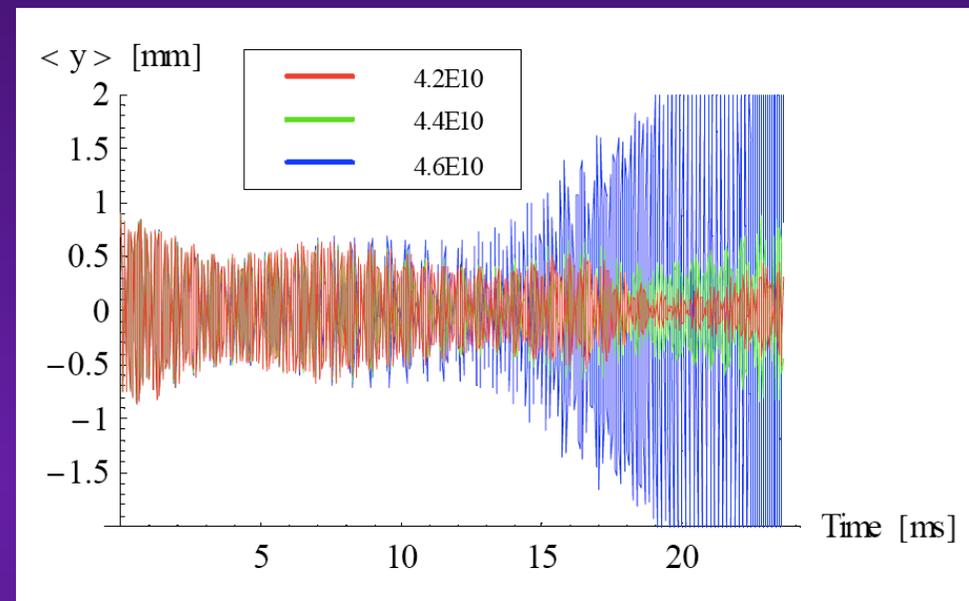
- The vertical Yokoya factor in the y-plane
- Slightly more than the horizontal Yokoya factor in the x-plane (it is not suppressed! and the effect of the detuning impedance, if any, seems small and in the plane of higher threshold)

# TMCI (40/40)

## WITHOUT LINEAR COUPLING



## WITH LINEAR COUPLING



$$Q_x = 26.180 \quad Q_y = 26.185 \quad \xi_{x,y} = 0$$

$$K_{skew}^{\text{HEADTAIL}} = 0.005 \text{ m}^{-1}$$

**=> The vertical intensity threshold is increased from  $\sim 3.3\text{E}10$  p/b to  $\sim 4.5\text{E}10$  p/b, i.e. an increase of 36%, in good agreement with a previous theoretical prediction of 33%**

## Transverse coupled-bunch instability in time domain (1/11)

- ◆ **The transverse coupled-bunch instability in circular machines is usually discussed using Sacherer's formula in the frequency domain**
- ◆ **Due to the periodicity of the machine, it can be derived for any wake-field in the case of equi-populated and equi-spaced bunches**
- ◆ **This formula takes into account the wake-field from all the preceding bunches and from all the previous turns. Furthermore, the intra-bunch motion is also taken into account. This approach is certainly still valid when the bunch train is much longer than the gap and for long-range wakes. However, when the gap is much larger than the train only a rough estimate can be expected**
- ◆ **In this case it is better to make a time-domain analysis, which is done in the following**
- ◆ **2 formulae will be proposed, the first for the case of the resistive-wall impedance with (or without) inductive bypass (i.e. taking into account the 1<sup>st</sup> low-frequency regime, which is of great importance for instance for the LHC collimators), and the second for the case of a resonator impedance**

## Transverse coupled-bunch instability in time domain (2/11)

- In the case of the resistive-wall impedance, the equation of motion for the bunch  $l$  (at azimuthal coordinate  $s$ ) submitted to the force exerted by the preceding bunch  $k$  (at azimuthal position  $s + z_k$ ) is given by, assuming first only the 2<sup>nd</sup> frequency regime (i.e. the classical thick wall regime, and considering macroparticle bunches

$$\frac{d^2 x_l(s)}{ds^2} + \left( \frac{Q_{x0}}{R} \right)^2 x_l(s) = \frac{F_{x,kl}}{\beta^2 E_{total}}$$

x- (dipolar) Yokoya factor

with

$$F_{x,kl} = q Q \frac{W_x^{TW}(z_k)}{2\pi R} x_k(s + z_k)$$

$$W_x^{TW}(z_k) = \frac{L \beta c F}{\pi b^3} \sqrt{\frac{Z_0 \rho \mu_r}{\pi}} \times \frac{1}{\sqrt{z_k}}$$

$$z_k = (l - k) s_{bunch}$$

It is not defined with a – here!

Bunch spacing

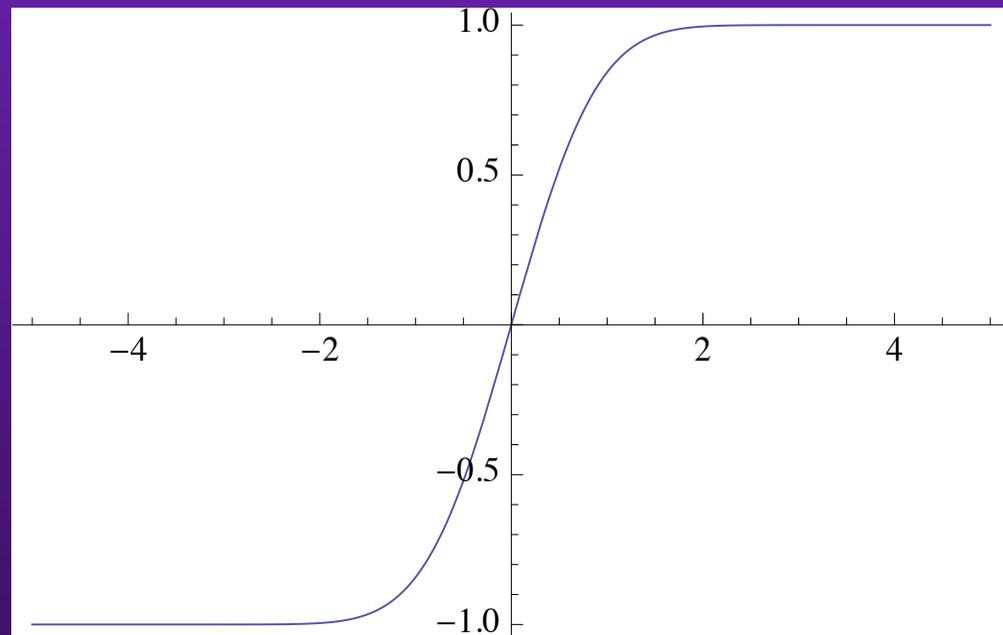
## Transverse coupled-bunch instability in time domain (3/11)

- ◆ In the presence of inductive bypass, an approximate formula for the wake field is given by (A. Koschik, 2003)

$$W_x^{IB}(z_k) = W_x^{TW}(z_k) + \frac{2L\beta c\mu_r F\rho}{\pi b^4} e^{\frac{4\mu_r\rho}{b^2\mu_0 c}z_k} \left( 1 - \text{Erf} \left[ \sqrt{\frac{4\mu_r\rho}{b^2\mu_0 c}} z_k \right] \right)$$

Error function

$$\text{Erf}[x] = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$



## Transverse coupled-bunch instability in time domain (4/11)

- ◆ Summing over all the bunches and all the previous revolutions yields

$$\frac{d^2 x_l(s)}{ds^2} + \left( \frac{Q_{x0}}{R} \right)^2 x_l(s) = \sum_{k=1}^M x_k(s) \times$$

$$\left\{ \begin{aligned} & \chi(l-k-1) \sum_{m=0}^{\infty} e^{j \frac{Q}{R} z_{km}} \left[ \frac{k_{RW}}{\sqrt{z_{km}}} - k_{IB} e^{\alpha_{IB} z_{km}} \left[ 1 - \text{Erf} \left( \sqrt{\alpha_{IB} z_{km}} \right) \right] \right] \\ & + \chi(k-l) \sum_{m=1}^{\infty} e^{j \frac{Q}{R} z_{km}} \left[ \frac{k_{RW}}{\sqrt{z_{km}}} - k_{IB} e^{\alpha_{IB} z_{km}} \left[ 1 - \text{Erf} \left( \sqrt{\alpha_{IB} z_{km}} \right) \right] \right] \end{aligned} \right\}$$

with

$$k_{RW} = \frac{L}{2\pi R} \times \frac{N_b e^2 F}{p \pi^{3/2} b^3} \sqrt{Z_0 \rho \mu_r}$$

$$k_{IB} = \frac{L}{2\pi R} \times \frac{2 N_b e^2 F \rho \mu_r}{p \pi b^4}$$

$$\alpha_{IB} = \frac{4 \rho \mu_r}{b^2 Z_0}$$

## Transverse coupled-bunch instability in time domain (5/11)

- ◆ Here,  $x_l = X_l e^{j s Q / R}$  = transverse position of bunch  $l$

$$\Rightarrow \frac{d^2 x_l(s)}{ds^2} + \left( \frac{Q_{x0}}{R} \right)^2 x_l(s) \approx \frac{2 Q_{x0}}{R^2} (Q_{x0} - Q) x_l(s)$$

$$\chi(y) = 1 \quad \text{if } y \geq 0, \quad 0 \quad \text{otherwise}$$

$$z_{km} = (l - k) s_{bunch} + m 2\pi R \quad = \text{distance between bunch } l \text{ and } k$$

- ◆ This leads to an eigenvalue problem, which can then be solved numerically: from the imaginary part of the eigenvalues the instability rise-time can be computed

## Transverse coupled-bunch instability in time domain (6/11)

$$\left[ Q - Q_{x0} + \frac{R^2}{2 Q_{x0}} \sum_{m=1}^{\infty} e^{j m 2 \pi Q} \left\{ \frac{k_{RW}}{\sqrt{m 2 \pi R}} - k_{IB} e^{\alpha_{IB} m 2 \pi R} \left[ 1 - \text{Erf} \left( \sqrt{\alpha_{IB} m 2 \pi R} \right) \right] \right\} \right] x_l(s)$$

$$+ \frac{R^2}{2 Q_{x0}} \sum_{\substack{k=1 \\ k \neq l}}^M x_k(s) \left\{ \begin{aligned} & \chi(l-k-1) \sum_{m=0}^{\infty} e^{j \frac{Q}{R} z_{km}} \left[ \frac{k_{RW}}{\sqrt{z_{km}}} - k_{IB} e^{\alpha_{IB} z_{km}} \left[ 1 - \text{Erf} \left( \sqrt{\alpha_{IB} z_{km}} \right) \right] \right] \\ & + \chi(k-l-1) \sum_{m=1}^{\infty} e^{j \frac{Q}{R} z_{km}} \left[ \frac{k_{RW}}{\sqrt{z_{km}}} - k_{IB} e^{\alpha_{IB} z_{km}} \left[ 1 - \text{Erf} \left( \sqrt{\alpha_{IB} z_{km}} \right) \right] \right] \end{aligned} \right\} = 0$$

## Transverse coupled-bunch instability in time domain (7/11)

- ◆ In the case of a resonator impedance, the equation of motion for the bunch  $l$  (at azimuthal coordinate  $s$ ) submitted to the force exerted by the preceding bunch  $k$  (at azimuthal position  $s + z_k$ ) is given by (considering macroparticle bunches)

$$\frac{d^2 x_l(s)}{ds^2} + \left( \frac{Q_{x0}}{R} \right)^2 x_l(s) = \frac{F_{x,kl}}{\beta^2 E_{total}}$$

with

$$F_{x,kl} = q Q \frac{W_x^R(z_k)}{2\pi R} x_k(s + z_k)$$

$$W_x^R(z_k) = F \frac{\omega_R^2 R_x}{Q_R \bar{\omega}_R} e^{-\frac{\alpha}{c} z_k} \sin\left(\bar{\omega}_R \frac{z_k}{c}\right)$$

$$z_k = (l - k) s_{bunch}$$

**x- (dipolar) Yokoya factor**

## Transverse coupled-bunch instability in time domain (8/11)

- ◆ Summing over all the bunches and all the previous revolutions yields

$$\frac{d^2 x_l(s)}{ds^2} + \left( \frac{Q_{x0}}{R} \right)^2 x_l(s) = k_R \sum_{k=1}^M x_k(s) \times$$

$$\left\{ \begin{array}{l} \chi(l-k-1) \sum_{m=0}^{\infty} e^{j \frac{Q}{R} z_{km}} e^{-\frac{\alpha}{c} z_{km}} \sin \left[ \frac{\bar{\omega}_R}{c} z_{km} \right] \\ + \chi(k-l) \sum_{m=1}^{\infty} e^{j \frac{Q}{R} z_{km}} e^{-\frac{\alpha}{c} z_{km}} \sin \left[ \frac{\bar{\omega}_R}{c} z_{km} \right] \end{array} \right\}$$

with

$$k_R = \frac{1}{2\pi R} \times \frac{N_b e^2 F \omega_R^2 R_x}{p c Q_R \bar{\omega}_R}$$

$$\bar{\omega}_R = \omega_R \sqrt{1 - \frac{1}{4Q_R^2}}$$

$$\alpha = \frac{\omega_R}{2Q_R}$$

## Transverse coupled-bunch instability in time domain (9/11)

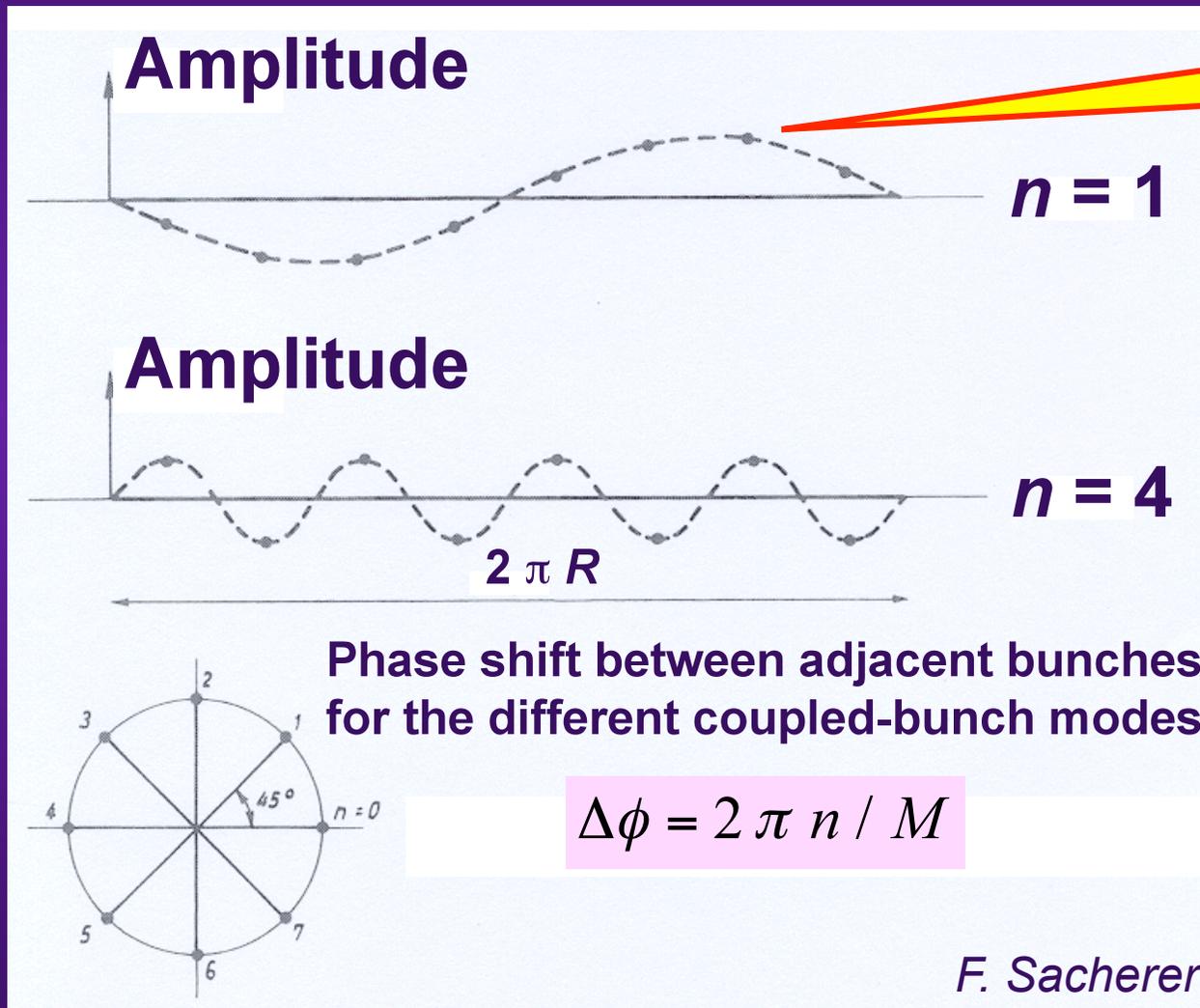
- ◆ This leads to an eigenvalue problem, which can then be solved numerically: from the imaginary part of the eigenvalues the instability rise-time can be computed

$$\left[ Q - Q_{x0} + \frac{R^2}{2 Q_{x0}} k_R \sum_{m=1}^{\infty} e^{j m 2 \pi Q} e^{-\frac{\alpha}{c} 2 \pi R m} \sin \left[ \frac{\bar{\omega}_R}{c} 2 \pi R m \right] \right] x_l(s)$$

$$+ \frac{R^2}{2 Q_{x0}} k_R \sum_{\substack{k=1 \\ k \neq l}}^M x_k(s) \left\{ \begin{array}{l} \chi(l-k-1) \sum_{m=0}^{\infty} e^{j \frac{Q}{R} z_{km}} e^{-\frac{\alpha}{c} z_{km}} \sin \left[ \frac{\bar{\omega}_R}{c} z_{km} \right] \\ + \chi(k-l-1) \sum_{m=1}^{\infty} e^{j \frac{Q}{R} z_{km}} e^{-\frac{\alpha}{c} z_{km}} \sin \left[ \frac{\bar{\omega}_R}{c} z_{km} \right] \end{array} \right\} = 0$$

# Transverse coupled-bunch instability in time domain (10/11)

## General picture in the case of equi-populated equi-spaced bunches



Bunch treated as a Macro-Particle

$M = 8$  bunches  $\Rightarrow$   
8 modes  $n$  (0 to 7)  
possible

**Reminder: 2 possible modes with 2 bunches (in phase or out of phase)**

## Transverse coupled-bunch instability in time domain (11/11)

**=> See the Movie for the case of a CERN SPS batch of 72 bunches  
(under Windows!)**